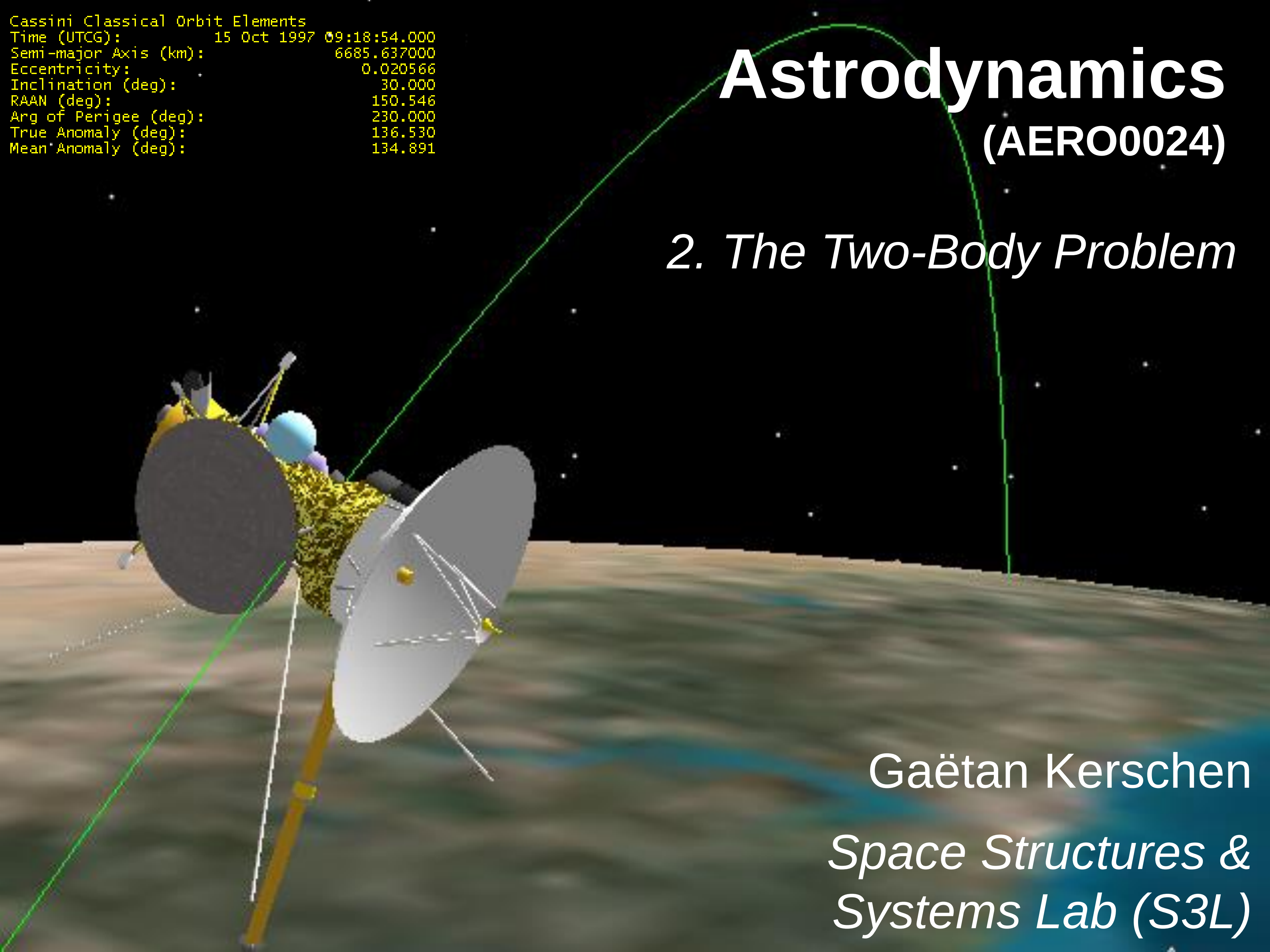
Astrodynamics
(AERO0024)

2. The Two-Body Problem



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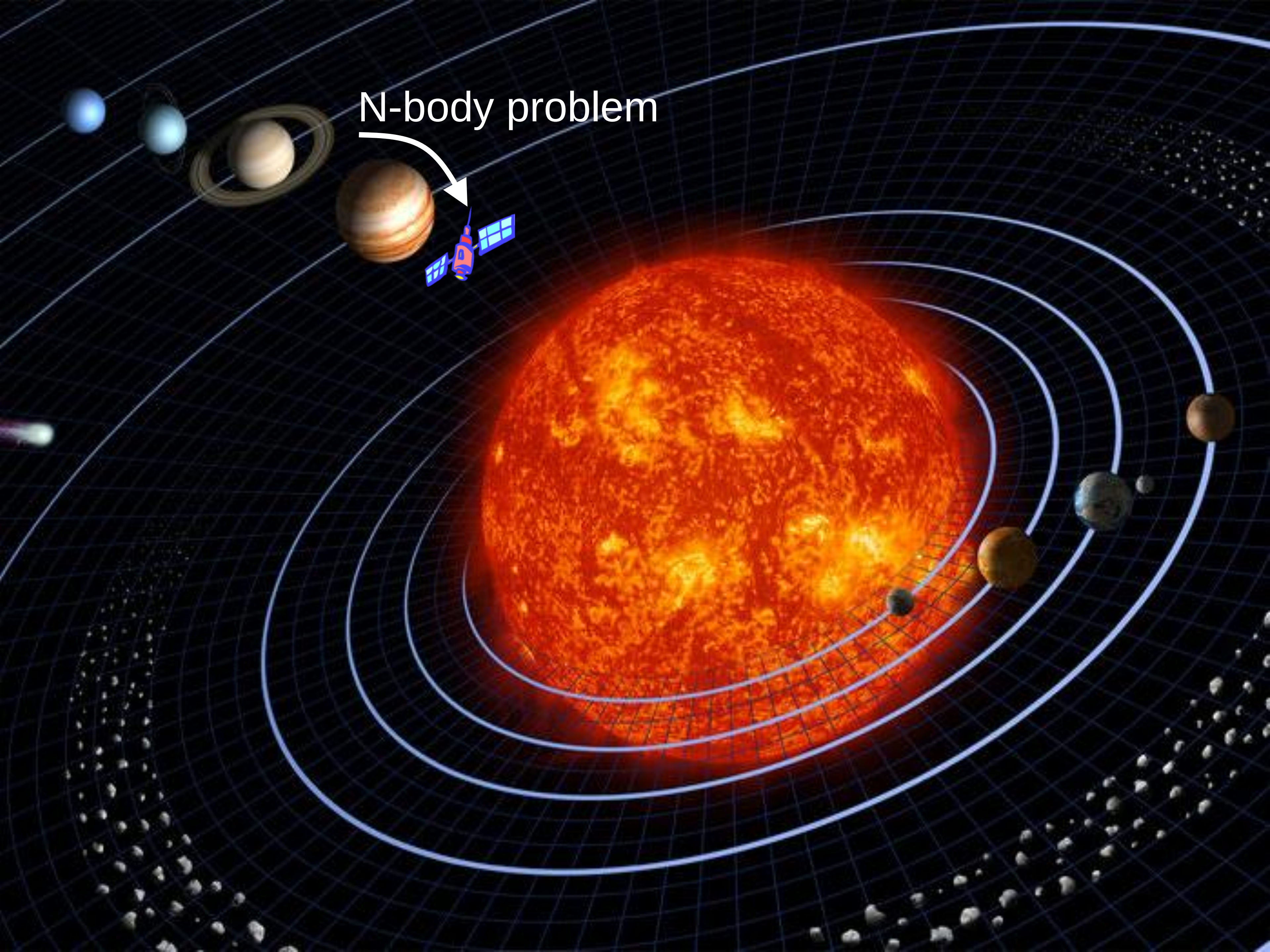

N-body problem

Interest in the Two-Body Problem ?

Precise orbit propagation:

Elaborate models are necessary to compute the motion of satellites to the high level of accuracy required for many applications today (e.g., the GPS system). The 2-body problem is not helpful in that context.

Interest in the Two-Body Problem ?

Qualitative understanding:
The main features of satellite and planet orbits can be described by a reasonably simple approximation, the two-body problem.

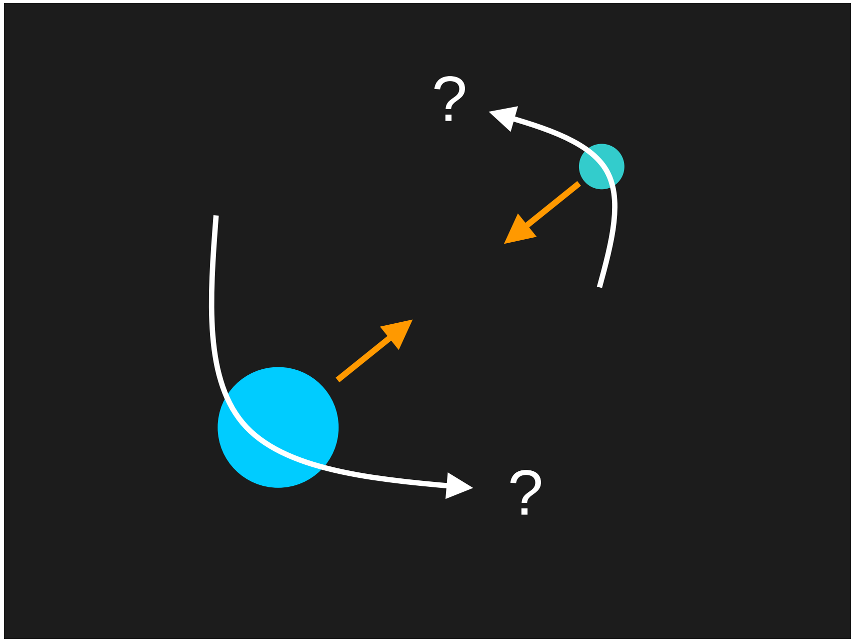
Mission design:
Some important quantities (ΔV and C_3) can be computed fairly accurately using the two-body assumption.

Interplanetary transfer:
In lecture 6, we will use a sequence of 2-body problems to approximate a complex interplanetary mission.

Definition of the 2-Body Problem

Motion of two bodies due solely to their own mutual gravitational attraction. Also known as **Kepler problem**.

Assumption: two point masses (or equivalently spherically symmetric objects).



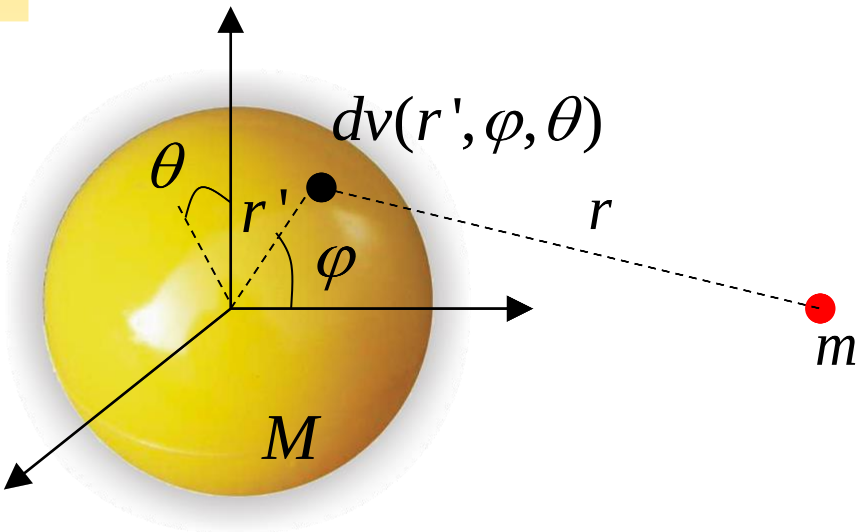
Bodies with Spatial Extent

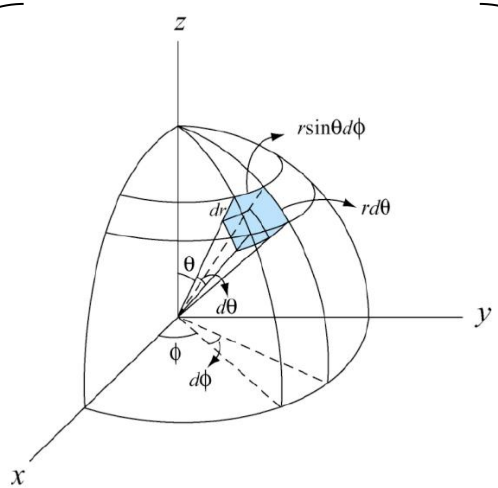
Up to now, point masses were considered.

But an object with a spherically-symmetric distribution of mass exerts the same gravitational attraction on external bodies as if all the object's mass were concentrated at a point at its centre.

Point mass M   Sphere of mass M

Spherically Symmetric Mass Distribution





$$dV = r^2 \sin\theta \, d\theta \, d\phi \, dr$$

$$M = \iiint_v \rho dv$$

$$V = -Gm \iiint_v \frac{\rho dv}{r}$$

$$dv = r'^2 \sin\varphi \, \mathrm{d}\varphi \, \mathrm{d}\theta \, \mathrm{d}r'$$

$$r = \sqrt{R^2 + r'^2 - 2r' R \cos\varphi}$$

$$\frac{dr}{d\varphi} = \frac{r' R \sin\varphi}{r}$$

Spherically Symmetric Mass Distribution

$$M = \left(\int_0^{2\pi} \mathrm{d}\theta\right)\left(\int_0^{\pi} \sin\varphi \,\mathrm{d}\varphi\right)\left(\int_0^{R_0} \rho r'^2 \,\mathrm{d}r'\right) = 4\pi\left(\int_0^{R_0} \rho r'^2 \,\mathrm{d}r'\right)$$

$$V = -2\pi G m\left(\int_0^{R_0}\left(\int_0^{\pi} \frac{\sin\varphi \,\mathrm{d}\varphi}{r}\right)\rho r'^2 \,\mathrm{d}r'\right)$$

$$= -2\pi G m\left(\int_0^{R_0}\left(\frac{1}{r' R}\int_{R-r'}^{R+r'} \mathrm{d}r\right)\rho r'^2 \,\mathrm{d}r'\right)$$

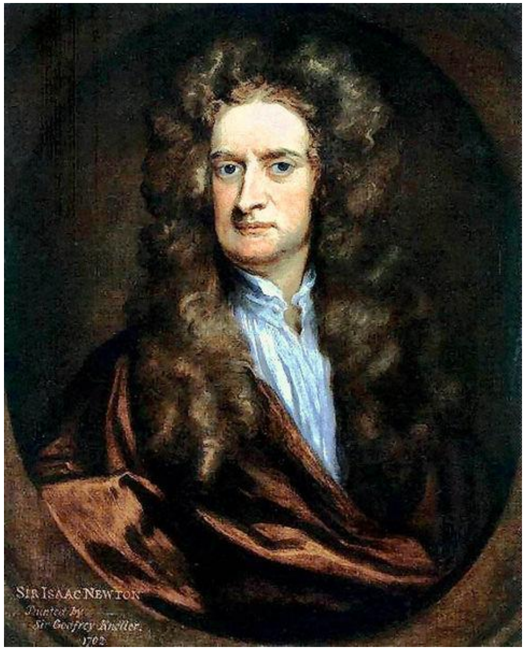
$$= -\frac{4\pi G m}{R}\left(\int_0^{R_0} \rho r'^2 \,\mathrm{d}r'\right) = -\frac{GMm}{R}$$

OK !

Gravitational Force

The law of universal gravitation is an empirical law describing the gravitational attraction between bodies with mass.

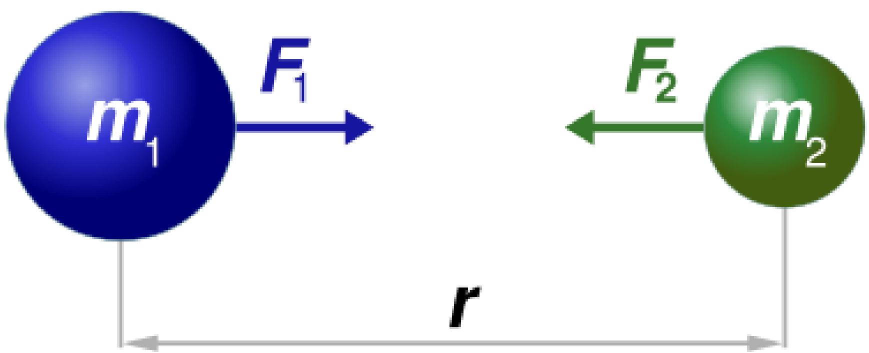
It was first formulated by Newton in *Philosophiae Naturalis Principia Mathematica* (1687). He was able to relate objects falling on the Earth to the motion of the planets.



Isaac Newton (1642-1727)

Gravitational Force

Every point mass attracts every other point mass by a force pointing along the line intersecting both points. The force is proportional to the product of the two masses and inversely proportional to the square of the distance between the point masses:



$$F_1 = F_2 = G\frac{m_1 \times m_2}{r^2}$$

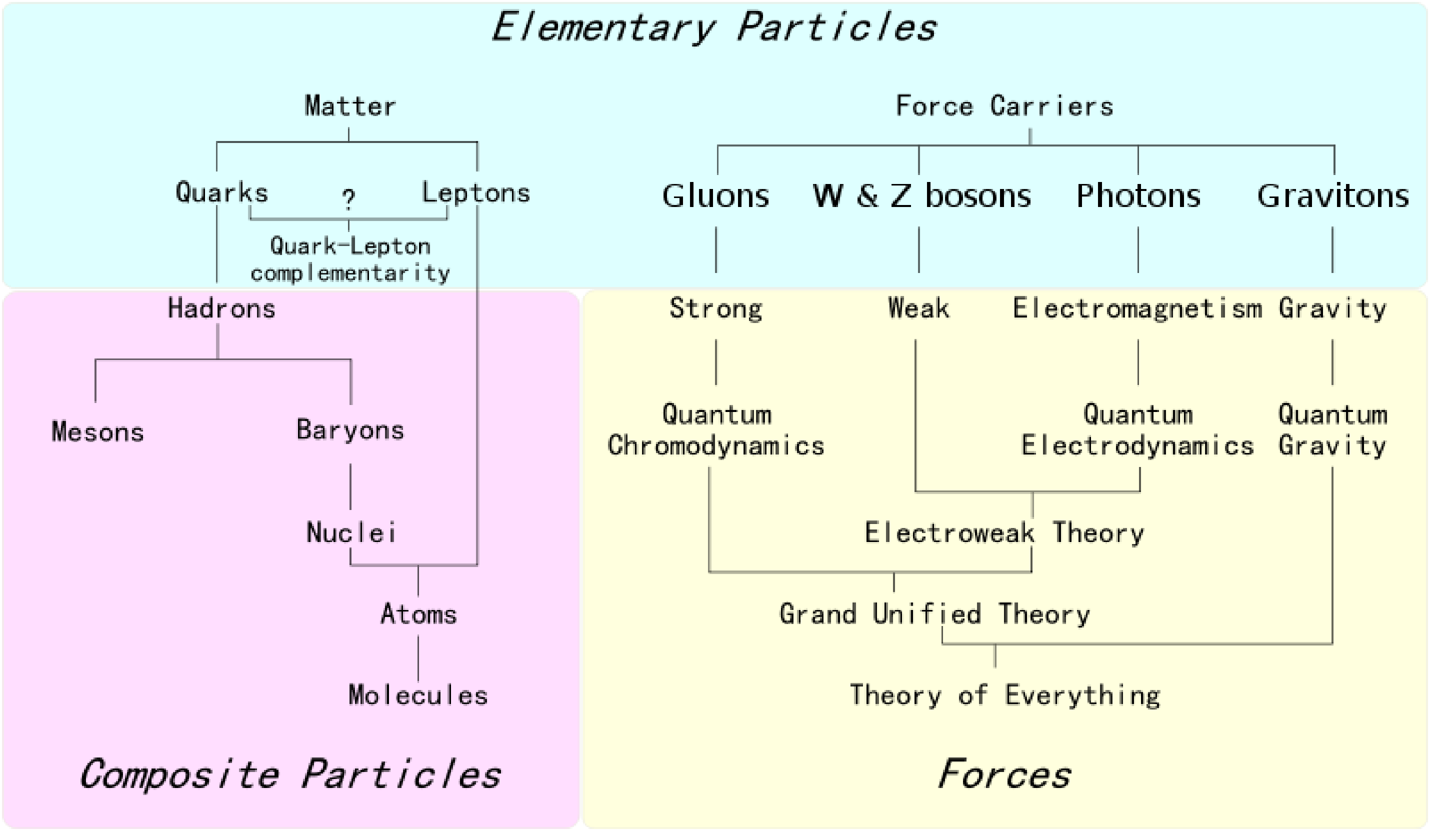
The Quest of a Unifying Theory

What is the relationship between the gravitational force and other known fundamental forces ?

That one body may act upon another at a distance through a vacuum without the mediation of anything else, by and through which their action and force may be conveyed from one another, is to me so great an absurdity that, I believe, no man who has in philosophic matters a competent faculty of thinking could ever fall into it. (Newton, 1692)

The question is not yet fully resolved today !

The Quest of a Unifying Theory



Peter Higgs and François Englert were awarded the Nobel Prize in physics for their work in identifying and discovering the Higgs boson, the so-called "God particle" that could explain how the universe's elementary particles obtained their mass shortly after the Big Bang.

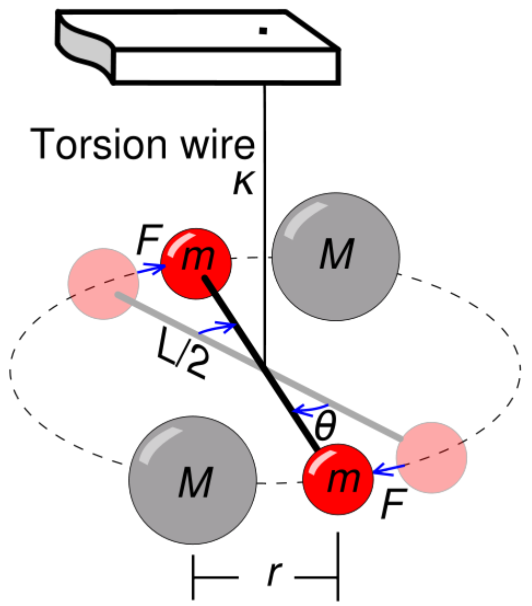
Gravitational Constant

By measuring the mutual attraction of two bodies of known mass, the gravitational constant *G* can directly be determined from torsion balance experiments.

Due to the small size of the gravitational force, *G* is presently only known with limited accuracy and was first determined many years after Newton's discovery:

$$(6.67428 \pm 0.00067) \times 10^{-11}\ m^3.kg^{-1}.s^{-2}$$

(http://www.physics.nist.gov/cgi-bin/cuu/Value?bg)



$$k\theta = L\frac{GmM}{r^2}$$

Gravitational Parameter of a Celestial Body

$$\mu = G M_{\oplus}$$

The gravitational parameter of the Earth has been determined with considerable precision from the analysis of laser distance measurements of artificial satellites:

$$398600.4418 \pm 0.0008 \text{ km}^3.\text{s}^{-2}.$$

The uncertainty is 1 to 5e8, much smaller than the uncertainties in *G* and *M* separately (~1 to 1e4 each).

Satellite Laser Ranging

TIGO (Concepcion, Chile)

LAGEOS-1

Lasers measure ranges from ground stations to satellite borne retro-reflectors. Because the events of sending and receiving a pulse can be registered within a few picoseconds, the distance between the ground station and the satellite is determined within a few millimeters.

Acceleration of Gravity

$$g = \frac{GM}{r^2}$$

$$g_{earth,SL} = 9.807\ \text{m/s}^2$$

$$g_{earth,aircraft} = g_{earth,SL} - 0.3\,\%$$

$$g_{earth,ISS} = g_{earth,SL} - 10\ \%$$

We sense our own weight by feeling contact forces acting on us in opposition to the force of gravity: W=mg.

If planetary gravity is the only force acting on a body, then the body is said to be in free fall. There are, by definition, no contact forces, so there can be no sense of weight.

A person in free fall experiences weightlessness: gravity is still there, but he cannot feel it.

2-Body Problem: Governing Equations

Newton's second law:

$F=ma$ where F is the gravitational force

2-Body Problem: Governing Equations

Newton's second law:

F=ma where *F* is the gravitational force

What did Richard Feynman mean about the Second Law of Motion? Where was the error?

JANUARY 17, 2021 / FRANCES48 / 0 COMMENTS

Richard Feynman writes about Newton's Second Law of Motion in his work "Lectures on Physics" (Chapter 15):

„For over 200 years the equations of motion enunciated by Newton were believed to describe nature correctly, and the first time that an error in these laws was discovered, the way to correct it was also discovered. Both the error and its correction were discovered by Einstein in 1905.

2-Body Problem: Governing Equations

Newton's Second Law, which we have expressed by the equation

$$F = d(mv)/dt$$

was stated with the tacit assumption that m is a constant, but we now know that this is not true, and that the mass of a body increases with velocity. In Einstein's corrected formula m has the value

$$m = \frac{m_0}{\sqrt{1 - v^2 / c^2}}$$

where the rest mass represents the mass of a body that is not moving and c is the speed of light [...].

Newton's law is still an excellent approximation of the effects of gravity if:

$$\frac{\Phi}{c^2} = \frac{GM}{rc^2} <<< 1, \text{ and } \left(\frac{v}{c}\right)^2 <<< 1$$

General Relativity: Earth-Sun Example

$$\frac{\Phi}{c^2} = \frac{GM_{sun}}{r_{orbit}c^2} \sim 10^{-8}, \text{ and } \left(\frac{v}{c}\right)^2 = \left(\frac{2\pi r_{orbit}}{1 \text{ year}.c}\right)^2 \sim 10^{-8}$$

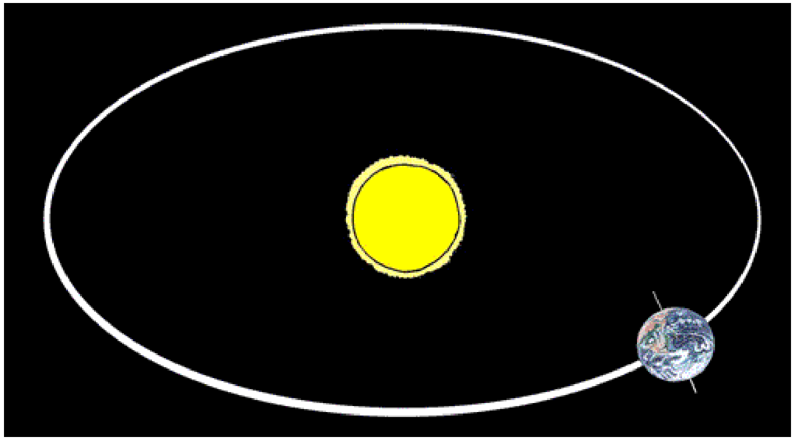
OK !

$G=6.67428 \times 10^{-11}\ m^3.kg^{-1}.s^{-2}$

$r_{orbit}=1.5 \times 10^{11}$ m (1 AU)

$M_{sun}=1.9891 \times 10^{30}$ kg

$c=3e8\ m.s^{-1}$



Motion of the Center of Mass

$$m_1\ddot{\mathbf{R}}_1 = \frac{Gm_1m_2}{r^2}\hat{\mathbf{u}}_r$$

$$+$$

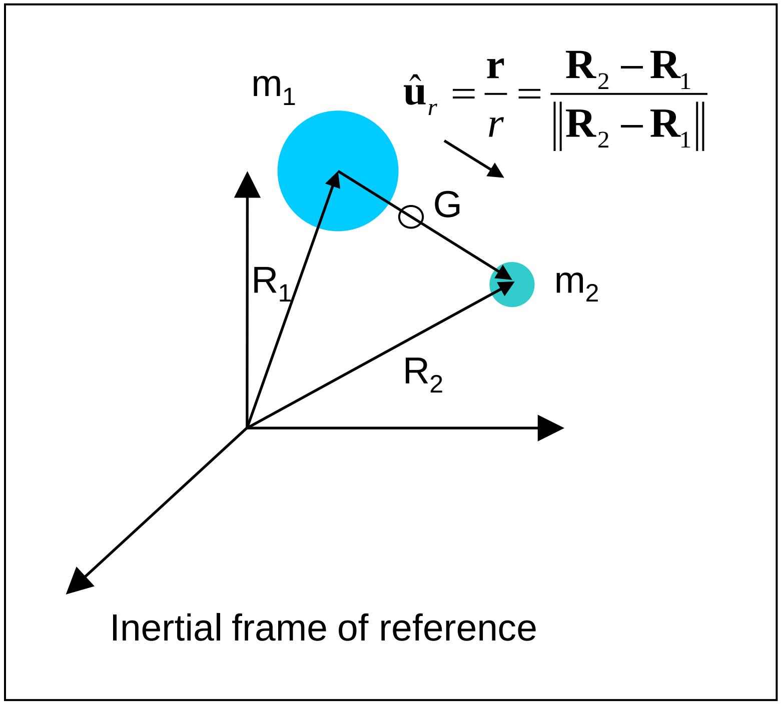
$$m_2\ddot{\mathbf{R}}_2 = -\frac{Gm_1m_2}{r^2}\hat{\mathbf{u}}_r$$

$$m_1\ddot{\mathbf{R}}_1 + m_2\ddot{\mathbf{R}}_2 = 0$$

$$+$$

$$\mathbf{R}_G = \frac{m_1\mathbf{R}_1 + m_2\mathbf{R}_2}{m_1 + m_2}$$

$$\mathbf{R}_G = \mathbf{R}_{G0} + \mathbf{v}_G t$$



⇒ The c.o.m. of a 2-body system may serve as the origin of an inertial frame.

Equations of Relative Motion

$$-m_1 m_2 \ddot{\mathbf{R}}_1 = \frac{-G m_1 m_2^2}{r^2} \hat{\mathbf{u}}_r$$

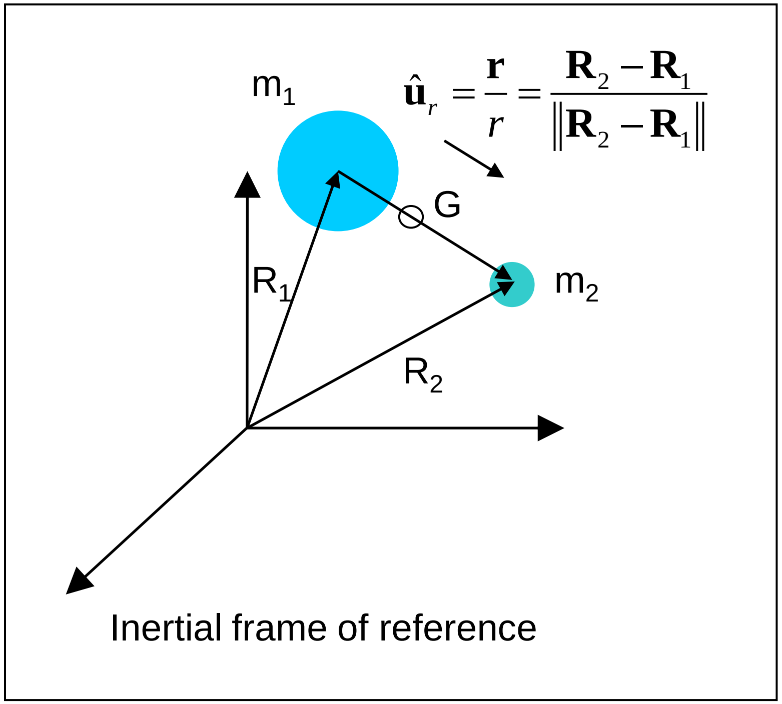
$$+$$

$$m_1 m_2 \ddot{\mathbf{R}}_2 = -\frac{G m_1^2 m_2}{r^2} \hat{\mathbf{u}}_r$$

$$\ddot{\mathbf{R}}_2 - \ddot{\mathbf{R}}_1 = -\frac{G(m_1 + m_2)}{r^2} \hat{\mathbf{u}}_r$$

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3} \mathbf{r}$$

μ is the gravitational parameter

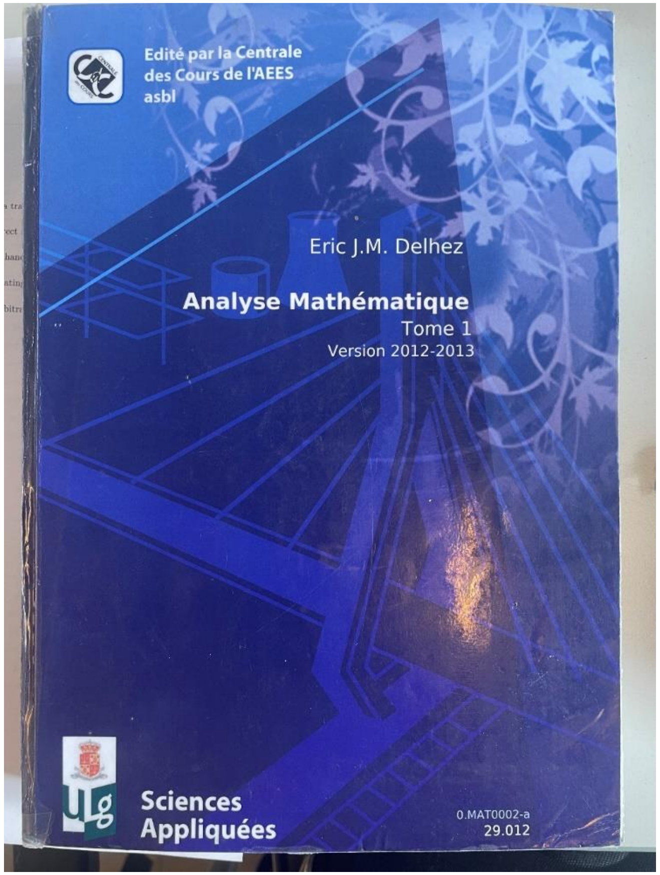


The motion of m_2 as seen from m_1 is the same as the motion of m_1 as seen from m_2.

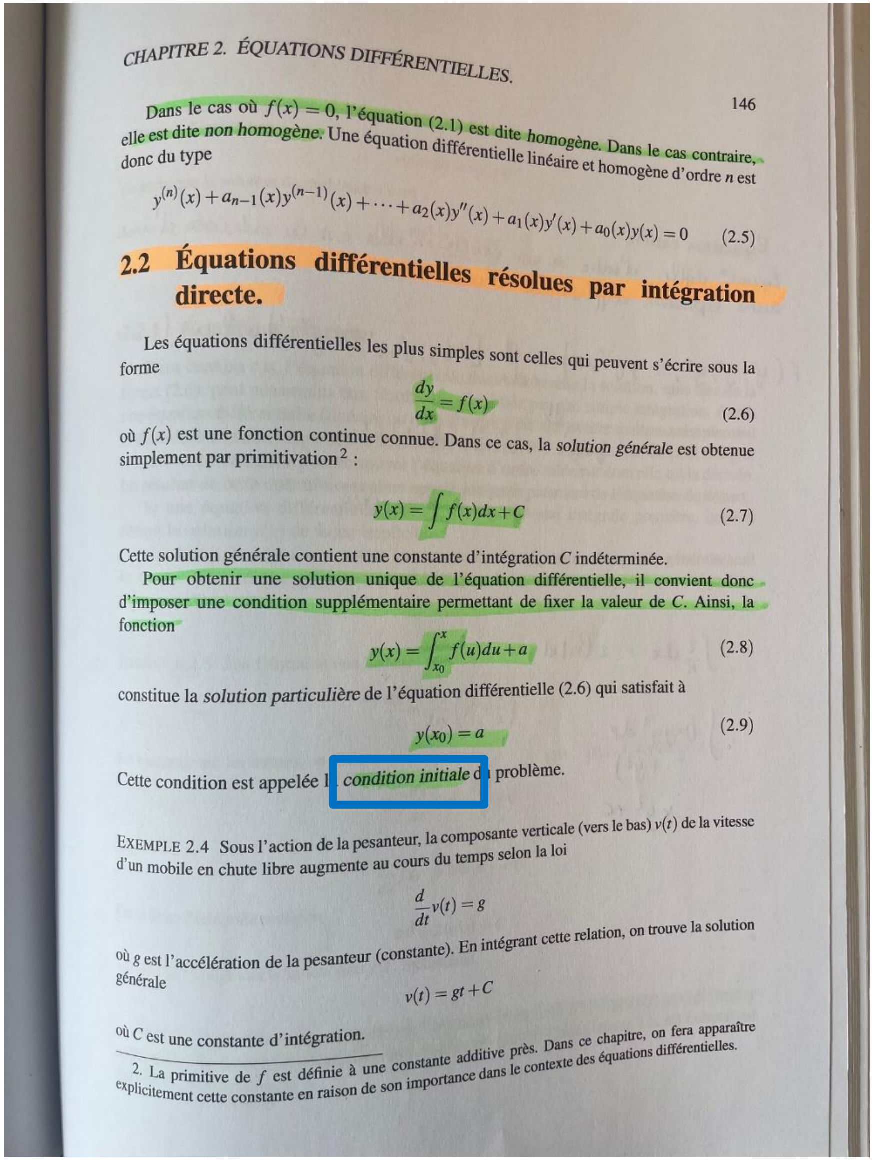
Equations of Relative Motion

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r}$$

How to solve it and find $\mathbf{r} = \mathbf{r}(t)$?



Equations of Relative Motion



Dans le cas où $f(x) = 0$, l'équation (2.1) est dite *homogène*. Dans le cas contraire, elle est dite *non homogène*. Une équation différentielle linéaire et homogène d'ordre n est donc du type

$$y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = 0 \qquad (2.5)$$

2.2 Équations différentielles résolues par intégration directe.

Les équations différentielles les plus simples sont celles qui peuvent s'écrire sous la forme

$$\frac{dy}{dx} = f(x) \qquad (2.6)$$

où $f(x)$ est une fonction continue connue. Dans ce cas, la *solution générale* est obtenue simplement par primitivation[2] :

$$y(x) = \int f(x)dx + C \qquad (2.7)$$

Cette solution générale contient une constante d'intégration C indéterminée.

Pour obtenir une solution unique de l'équation différentielle, il convient donc d'imposer une condition supplémentaire permettant de fixer la valeur de C. Ainsi, la fonction

$$y(x) = \int_{x_0}^{x} f(u)du + a \qquad (2.8)$$

constitue la *solution particulière* de l'équation différentielle (2.6) qui satisfait à

$$y(x_0) = a \qquad (2.9)$$

Cette condition est appelée la *condition initiale* du problème.

EXEMPLE 2.4 Sous l'action de la pesanteur, la composante verticale (vers le bas) $v(t)$ de la vitesse d'un mobile en chute libre augmente au cours du temps selon la loi

$$\frac{d}{dt}v(t) = g$$

où g est l'accélération de la pesanteur (constante). En intégrant cette relation, on trouve la solution générale

$$v(t) = gt + C$$

où C est une constante d'intégration.

2. La primitive de f est définie à une constante additive près. Dans ce chapitre, on fera apparaître explicitement cette constante en raison de son importance dans le contexte des équations différentielles.

Equations of Relative Motion

$$v(t) = v_0 + gt$$

2.2.1 Équations exactes.

Dans certains cas, l'équation différentielle dont on cherche la solution, sans être de la forme (2.6), peut néanmoins être résolue ou simplifiée par une simple intégration. Ainsi, une équation différentielle (linéaire ou non) d'ordre n est dite *exacte* si elle est simplement la dérivée d'une autre équation différentielle d'ordre $n-1$. Dans ce cas, on peut intégrer l'équation différentielle pour retrouver l'équation d'ordre inférieur dont elle est la dérivée. Le résultat de cette opération est alors appelé *intégrale première* de l'équation de départ.

Si une équation différentielle d'ordre un possède une intégrale première, celle-ci définit la solution $y(x)$ de façon implicite.

Une intégrale première contient une constante d'intégration et exprime généralement la conservation d'une grandeur caractéristique du système représenté par l'équation différentielle.

EXEMPLE 2.5 Soit l'équation non linéaire

$$\frac{dy}{dx} = \frac{-1}{2xy}\left(y^2 + \frac{2}{x}\right)$$

En réarrangeant les termes, on obtient

$$2xy\frac{dy}{dx} + y^2 + \frac{2}{x} = 0$$

on peut intégrer

soit

$$\frac{d}{dx}\left(xy^2 + 2\ln|x|\right) = 0$$

On a donc l'intégrale première

$$xy^2 + 2\ln|x| = C$$

qui définit implicitement la fonction $y(x)$ recherchée. ◇

Parfois, il est nécessaire de multiplier les deux membres de l'équation par un facteur approprié afin de rendre celle-ci exacte et d'en permettre l'intégration. Un tel facteur est appelé *facteur intégrant*.

Energy Conservation

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r}$$

$$\ddot{\mathbf{r}}.\dot{\mathbf{r}} = \frac{1}{2}\frac{d}{dt}(\dot{\mathbf{r}}.\dot{\mathbf{r}}) = \frac{1}{2}\frac{d}{dt}\left(\dot{r}^{\mathbf{2}}\right) = \frac{1}{2}\frac{d}{dt}\left(v^{\mathbf{2}}\right)$$

$$\mu\frac{\mathbf{r}.\dot{\mathbf{r}}}{r^3} = \mu\frac{r.\dot{r}}{r^3} = \mu\frac{\dot{r}}{r^2} = -\frac{d}{dt}\left(\frac{\mu}{r}\right)$$

$$\frac{v^2}{2} - \frac{\mu}{r} = E$$

Constant Angular Momentum

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r} \quad \Rightarrow \quad \mathbf{r}\times\ddot{\mathbf{r}} = \mathbf{r}\times\left(-\frac{\mu}{r^3}\mathbf{r}\right) = 0$$

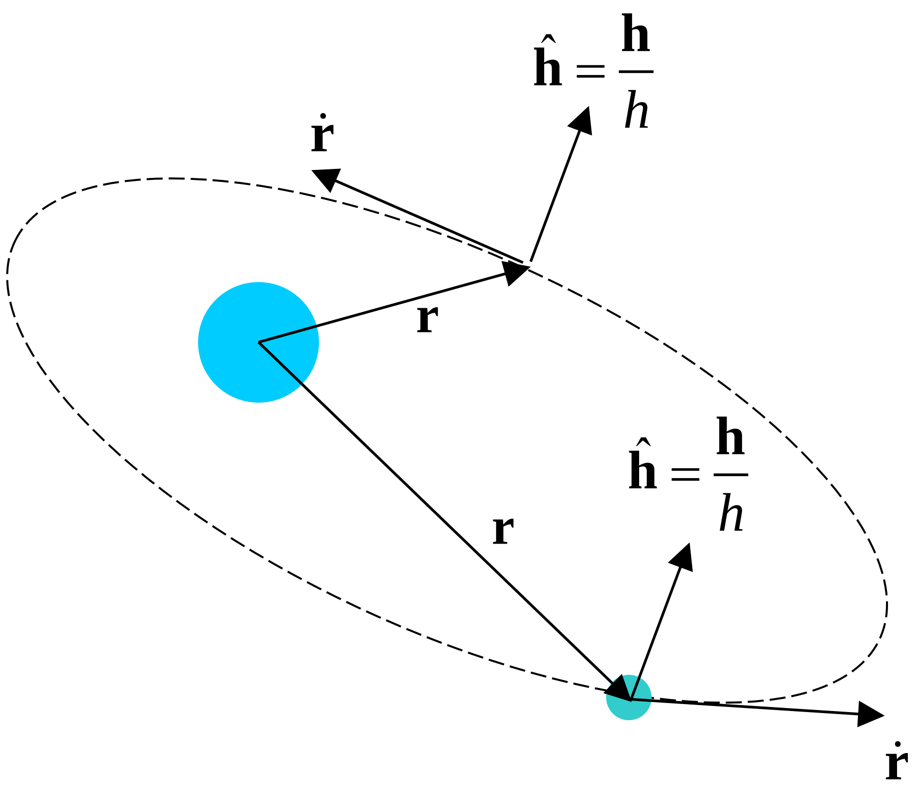
$$\mathbf{h} = \mathbf{r}\times\dot{\mathbf{r}} \quad \xrightarrow{d/dt} \quad \frac{d\mathbf{h}}{dt} = \dot{\mathbf{r}}\times\dot{\mathbf{r}} + \mathbf{r}\times\ddot{\mathbf{r}} = \mathbf{r}\times\ddot{\mathbf{r}}$$

Specific angular momentum

$$\Downarrow$$

$$\frac{d\mathbf{h}}{dt} = 0 \rightarrow \mathbf{r}\times\dot{\mathbf{r}} = \text{constant} = \mathbf{h}$$

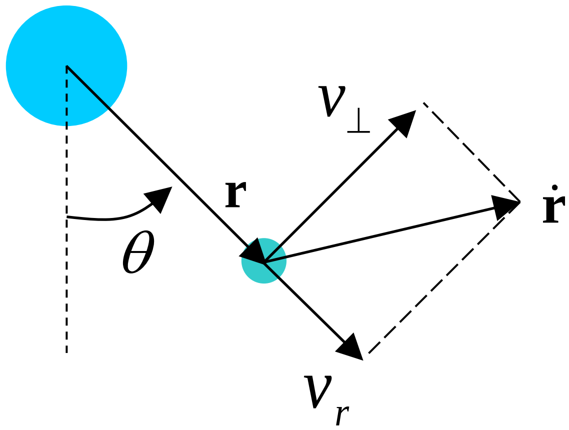
The Motion Lies in a Fixed Plane



The fixed plane is the **orbit plane** and is normal to the angular momentum vector.

$$\mathbf{r} \times \dot{\mathbf{r}} = \text{constant} = \mathbf{h}$$

Azimuth Component of the Velocity



$$\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}} = r\hat{\mathbf{u}}_r \times (v_r \hat{\mathbf{u}}_r + v_\perp \hat{\mathbf{u}}_\perp) = r v_\perp \hat{\mathbf{h}}$$

$$h = r v_\perp = r^2 \dot{\theta}$$

The angular momentum depends only on the azimuth component of the relative velocity

First Integral of Motion

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r} \quad \xRightarrow{\times \mathbf{h}} \quad \ddot{\mathbf{r}} \times \mathbf{h} = -\frac{\mu}{r^3}\mathbf{r} \times \mathbf{h} = -\frac{\mu}{r^3}\mathbf{r} \times (\mathbf{r} \times \dot{\mathbf{r}})$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a}.\mathbf{c}) - \mathbf{c}(\mathbf{a}.\mathbf{b}) \quad \Rightarrow \quad \ddot{\mathbf{r}} \times \mathbf{h} = \frac{\mu}{r^3}\left[\dot{\mathbf{r}}(\mathbf{r}.\mathbf{r}) - \mathbf{r}(\mathbf{r}.\dot{\mathbf{r}})\right]$$

$$\mathbf{r}.\dot{\mathbf{r}} = r\dot{r} \quad \Rightarrow \quad = \mu\left(\frac{\dot{\mathbf{r}}}{r} - \frac{\mathbf{r}\dot{r}}{r^2}\right) = \mu \frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right)$$

$$\int \quad \Rightarrow \quad \dot{\mathbf{r}} \times \mathbf{h} - \mu \frac{\mathbf{r}}{r} = \text{constant} = \mu \mathbf{e}$$

e lies in the orbit plane (***e***.**h**)=0: the line defined by **e** is the apse line. Its norm, *e*, is the eccentricity.

Note: demonstrate the Identity $\mathbf{r}.\dot{\mathbf{r}} = r\dot{r}$

$$\frac{d}{dt}(\mathbf{r}.\mathbf{r}) = \mathbf{r}.\frac{d\mathbf{r}}{dt} + \frac{d\mathbf{r}}{dt}.\mathbf{r} = 2\mathbf{r}.\frac{d\mathbf{r}}{dt} = 2\mathbf{r}.\dot{\mathbf{r}}$$

$$\mathbf{r}.\mathbf{r} = r^2 \quad \Rightarrow \quad \frac{d}{dt}(\mathbf{r}.\mathbf{r}) = 2r\frac{dr}{dt} = 2r\dot{r}$$

$$\Downarrow$$

$$\mathbf{r}.\dot{\mathbf{r}} = 2r\dot{r}$$

Orbit Equation

$$\frac{\dot{\mathbf{r}}\times\mathbf{h}}{\mu}=\frac{\mathbf{r}}{r}+\mathbf{e}$$

$\mathbf{r}.$ ⇒

$$\frac{\mathbf{r}.(\dot{\mathbf{r}}\times\mathbf{h})}{\mu}=\frac{\mathbf{r}.\mathbf{r}}{r}+\mathbf{r}.\mathbf{e}$$

$\mathrm{a}.(\mathrm{b}\times\mathrm{c})=(\mathrm{a}\times\mathrm{b}).\mathrm{c}$ ⇓

$$\frac{\mathbf{r}.(\dot{\mathbf{r}}\times\mathbf{h})}{\mu}=\frac{(\mathbf{r}\times\dot{\mathbf{r}}).\mathbf{h}}{\mu}=\frac{\mathbf{h}.\mathbf{h}}{\mu}=\frac{h^2}{\mu}=r+\mathbf{r}.\mathbf{e}$$

⇓

$$r=\frac{h^2}{\mu}\frac{1}{1+e\cos\theta}$$

Closed form of the nonlinear equations of motion

Conic Section in Polar Coordinates

Constant: angular momentum

Semi-latus rectum

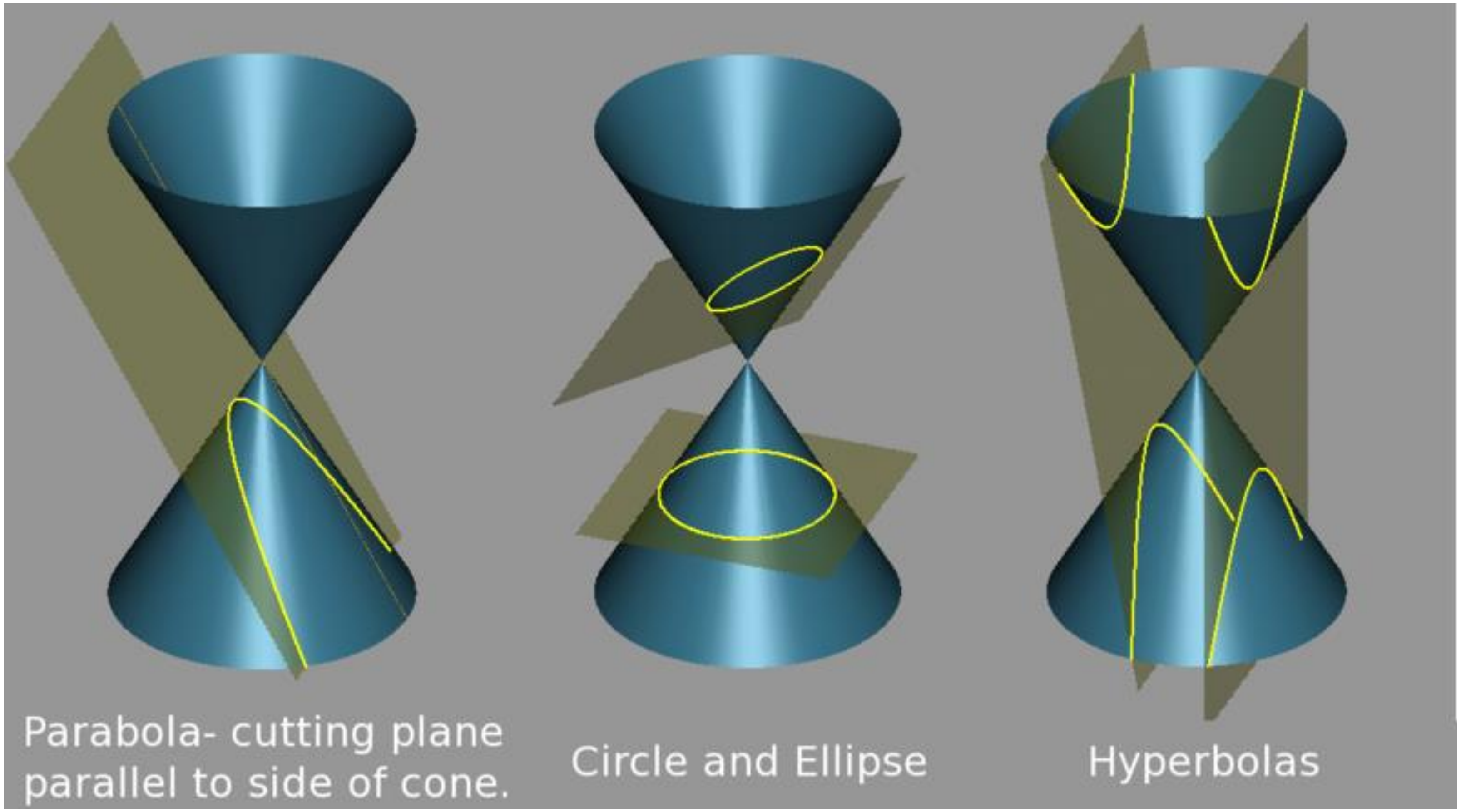
$$r = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta} = \frac{p}{1+e\cos\theta}$$

Constant: gravitational parameter

Independent variable: true anomaly (=0 at the periapsis)

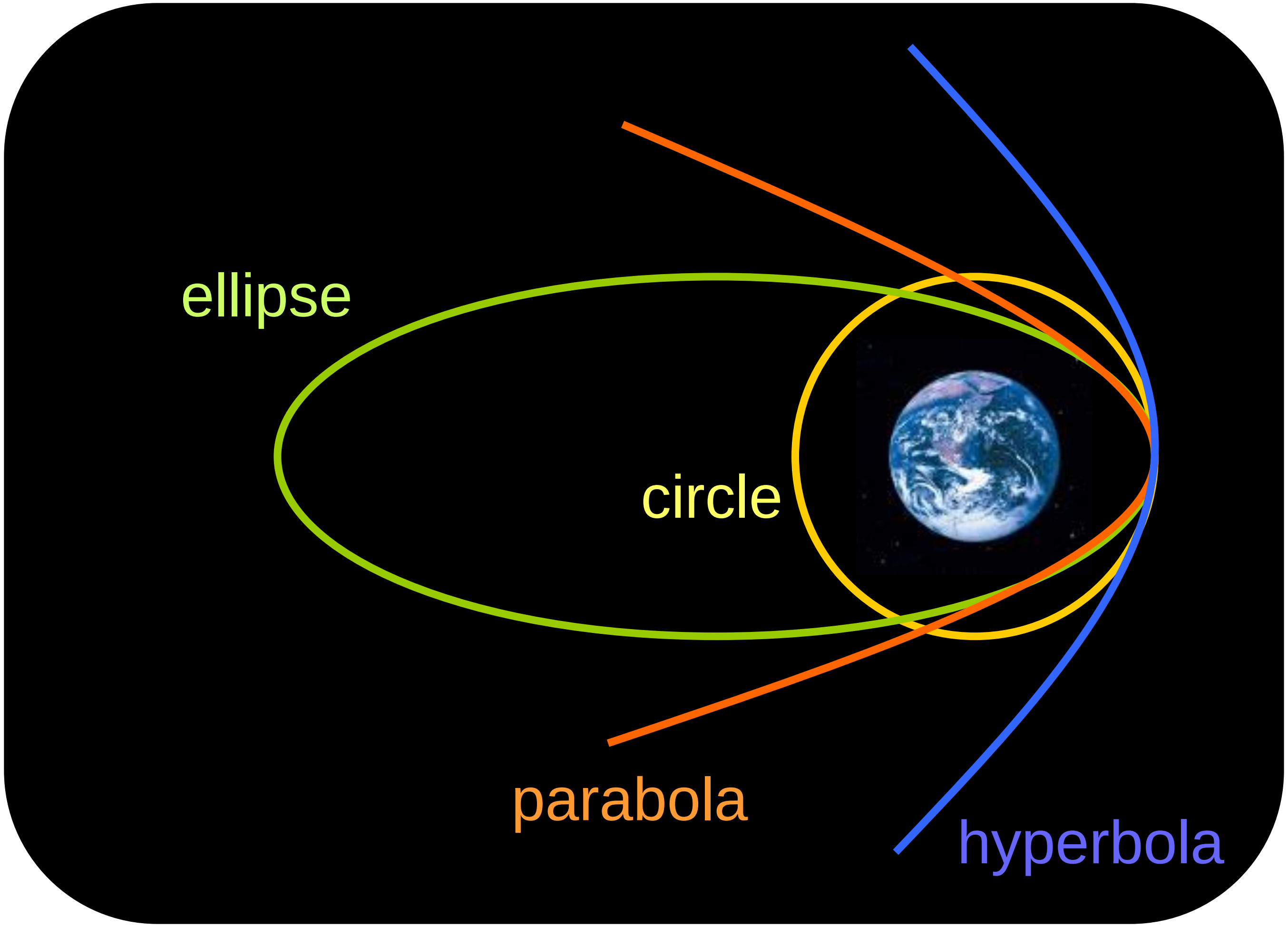
Constant: eccentricity

Conic Section



e=1 e=0 0<e<1 e>1

Possible Motions in the 2-Body System

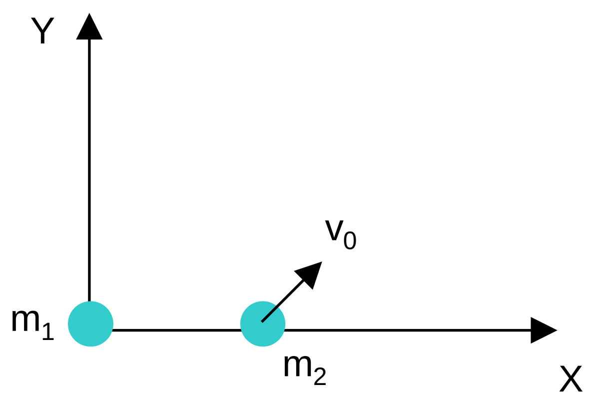


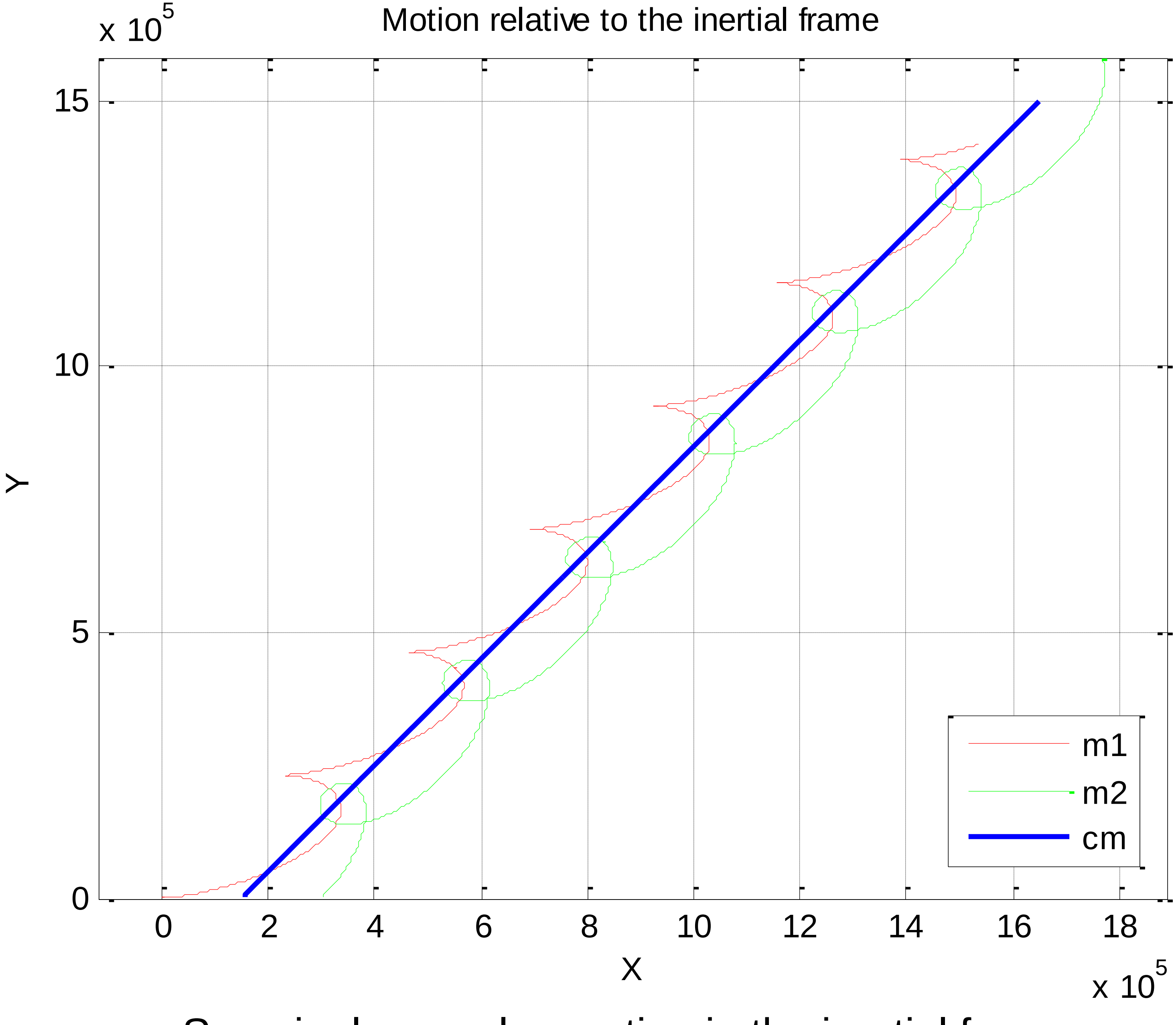
Two-Body Problem: Matlab Example

Two identical masses:

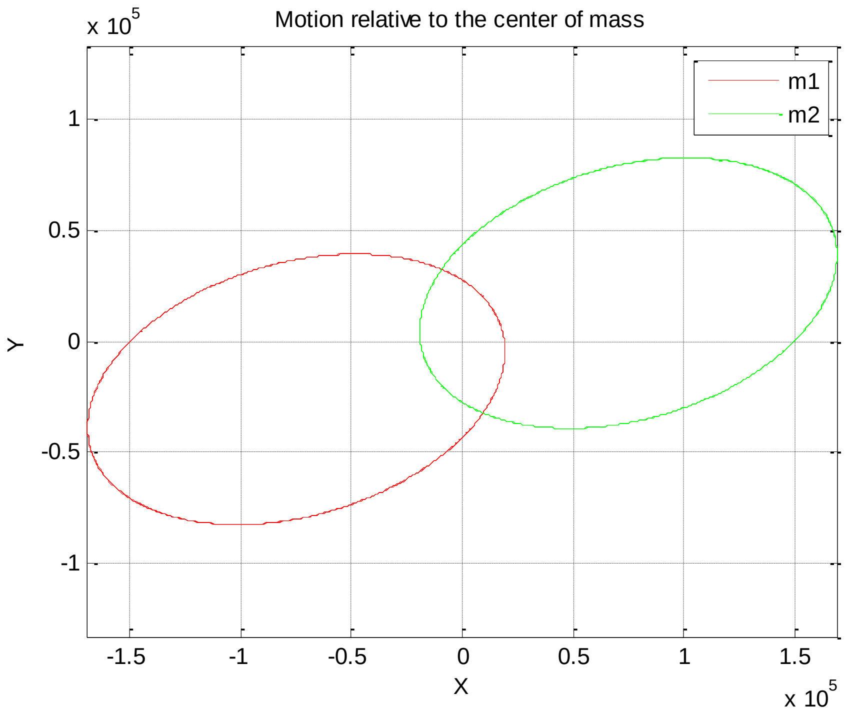
⇒ One is at rest at the origin of the inertial frame of reference.

⇒ The other one has a velocity directed upward to the right making a 45 degrees angle with the X axis.

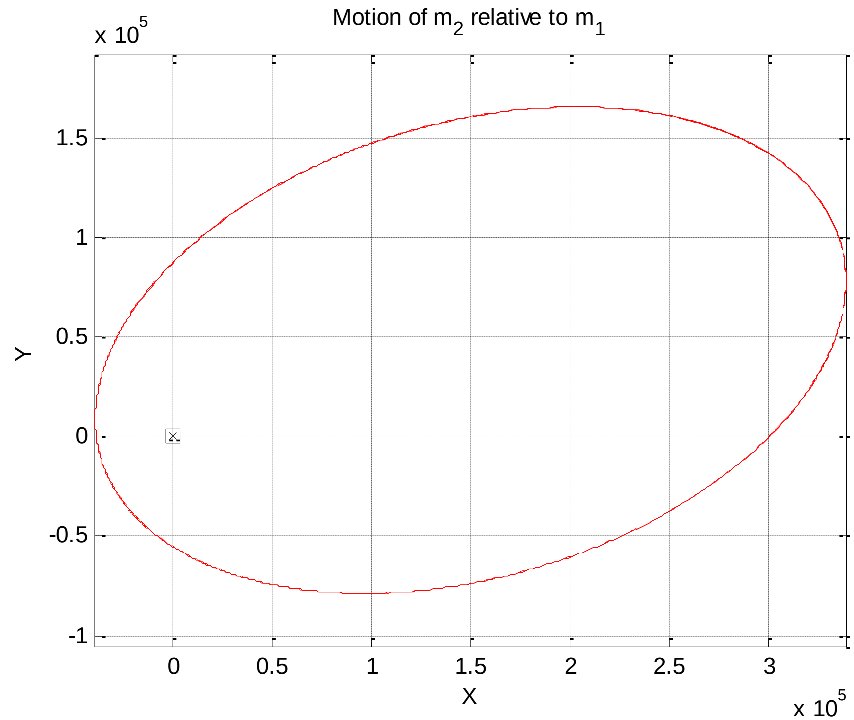




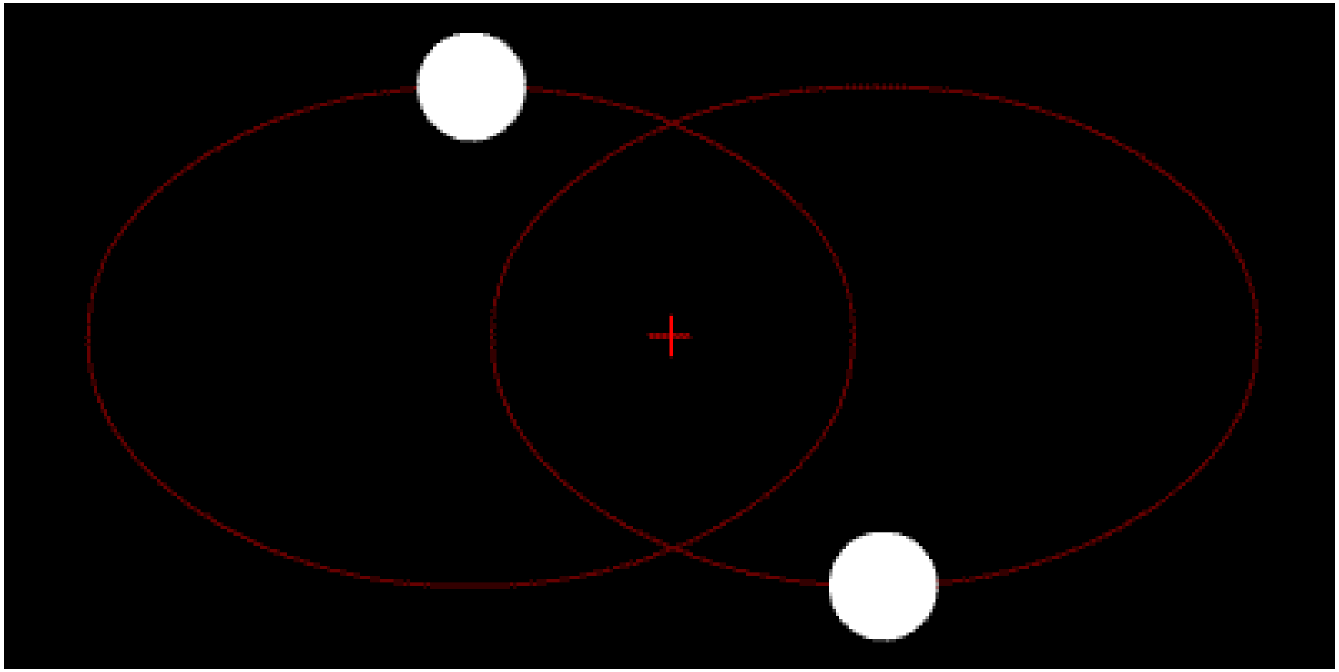
Seemingly complex motion in the inertial frame



Much less complex motion when viewed from the c.o.m



Much less complex motion when viewed from m_1



In Summary

- \+ We can calculate *r* for all values of the true anomaly.
- \+ The orbit equation is a mathematical statement of Kepler's first law.
- \- The solution of the "simple" problem of two bodies cannot be expressed in a closed form, explicit function of time.

Do we have 6 independent constants ?

The two vector constants **h** and **e** provide only 5 independent constants: **h.e**=0

Circular Orbits (e=0)

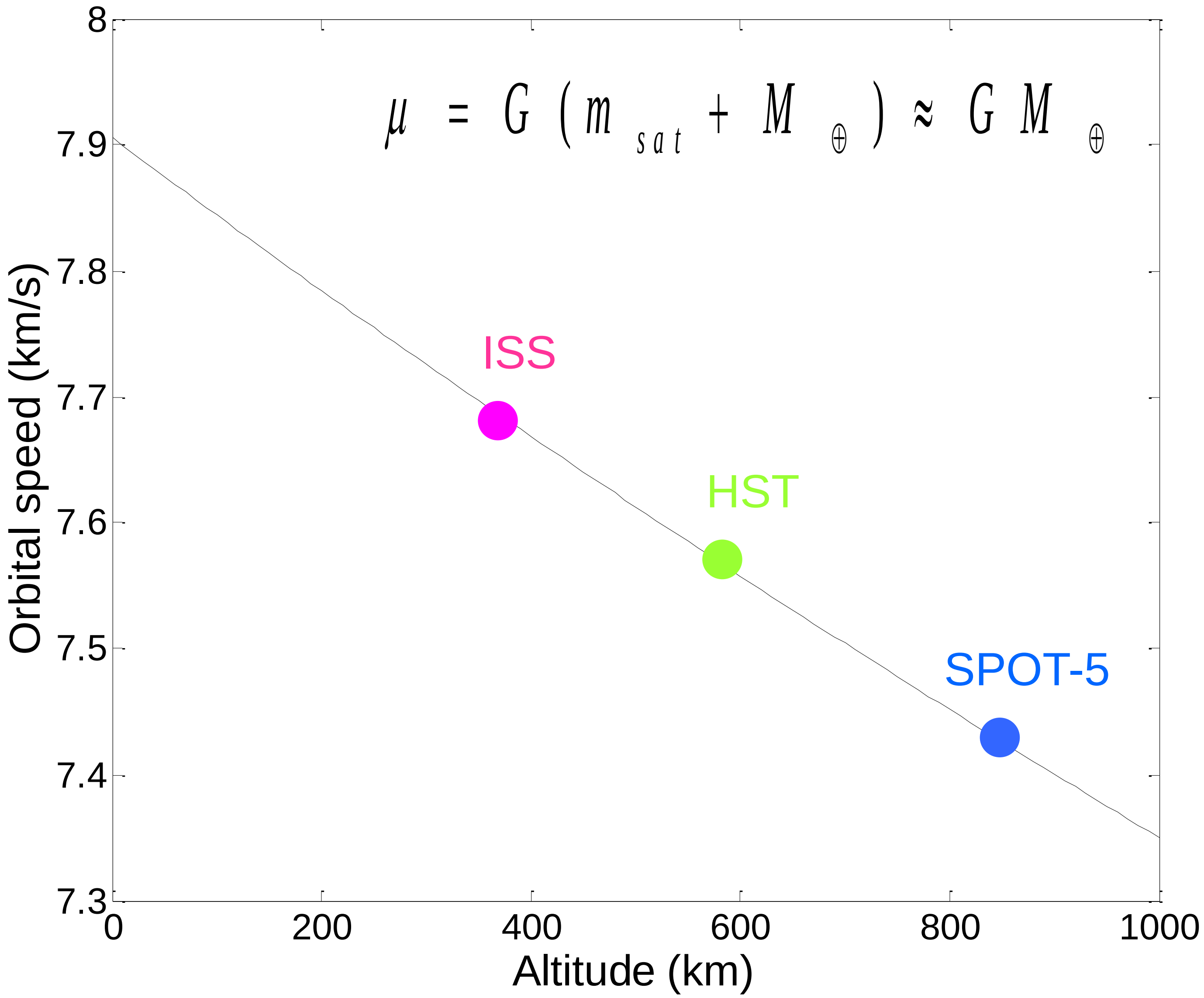
$$r = \frac{h^2}{\mu} = \text{Constant} \qquad h = rv_\perp = rv_{circular}$$

$$v_{circ} = \sqrt{\frac{\mu}{r}}$$

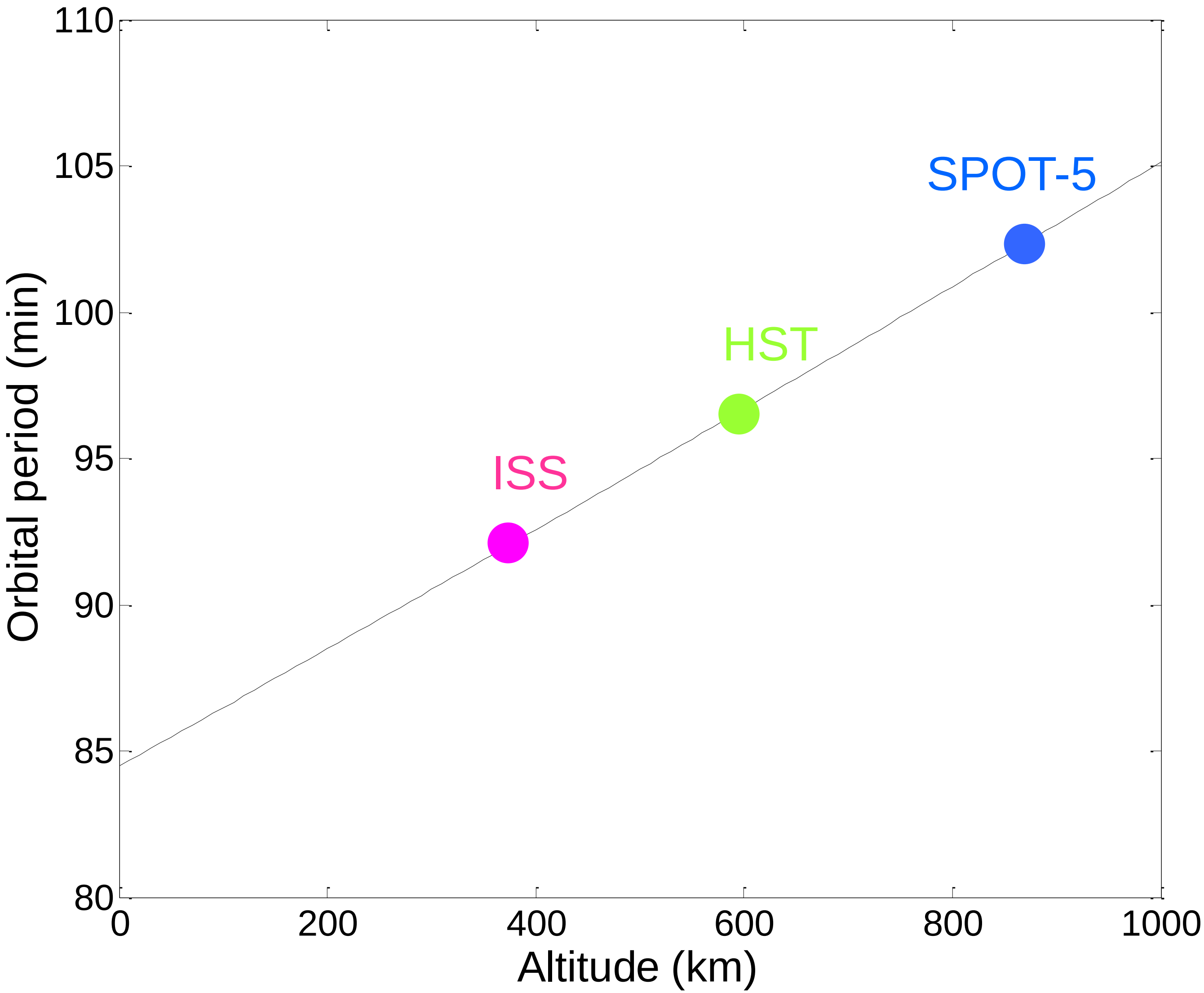
$$T_{circ} = 2\pi r \Big/ \sqrt{\frac{\mu}{r}} = \frac{2\pi}{\sqrt{\mu}} r^{3/2}$$

$$\varepsilon_{circ} = -\frac{\mu}{2r} < 0$$

Orbital Speed



Orbital Period



Two Particular Cases

1. 7.9 km/s is the **first cosmic velocity**; i.e., the minimum velocity (theoretical velocity, $r = 6378$ km) to orbit the Earth.

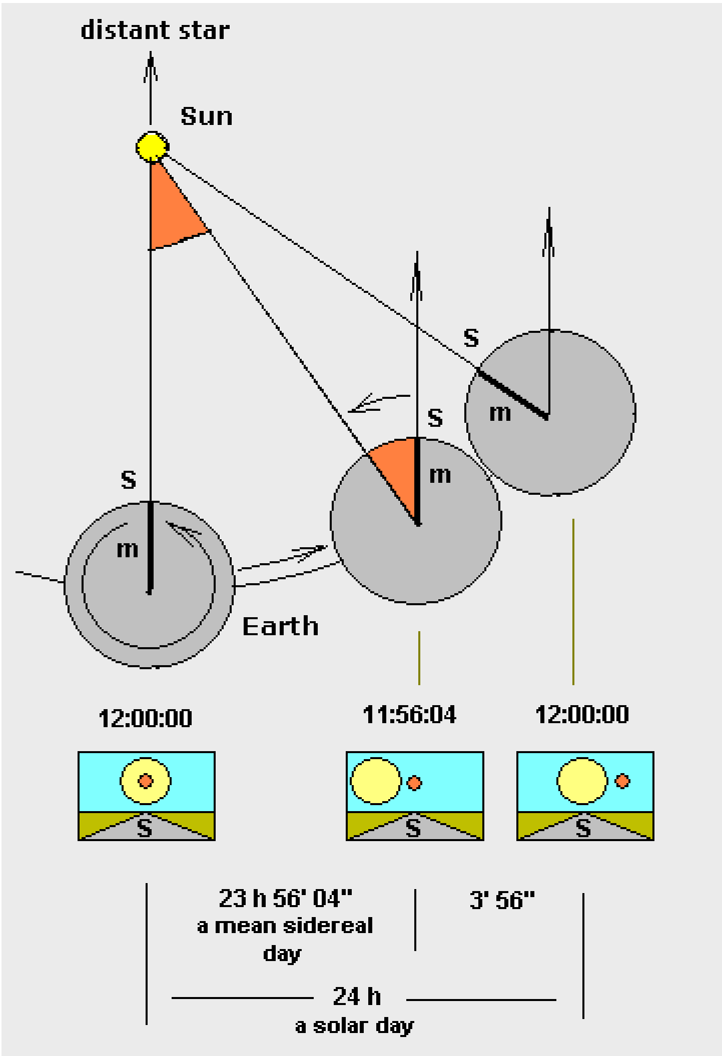
2. 35786 km is the altitude of the **geostationary orbit**. It is the orbit at which the satellite angular velocity is equal to that of the Earth, $\omega=\omega_E=7.292\ 10^{-5}$ rad/s, in inertial space (*).

$$r_{GEO=}\left(\frac{T_{circ}\sqrt{\mu}}{2\pi}\right)^{2/3}$$

---

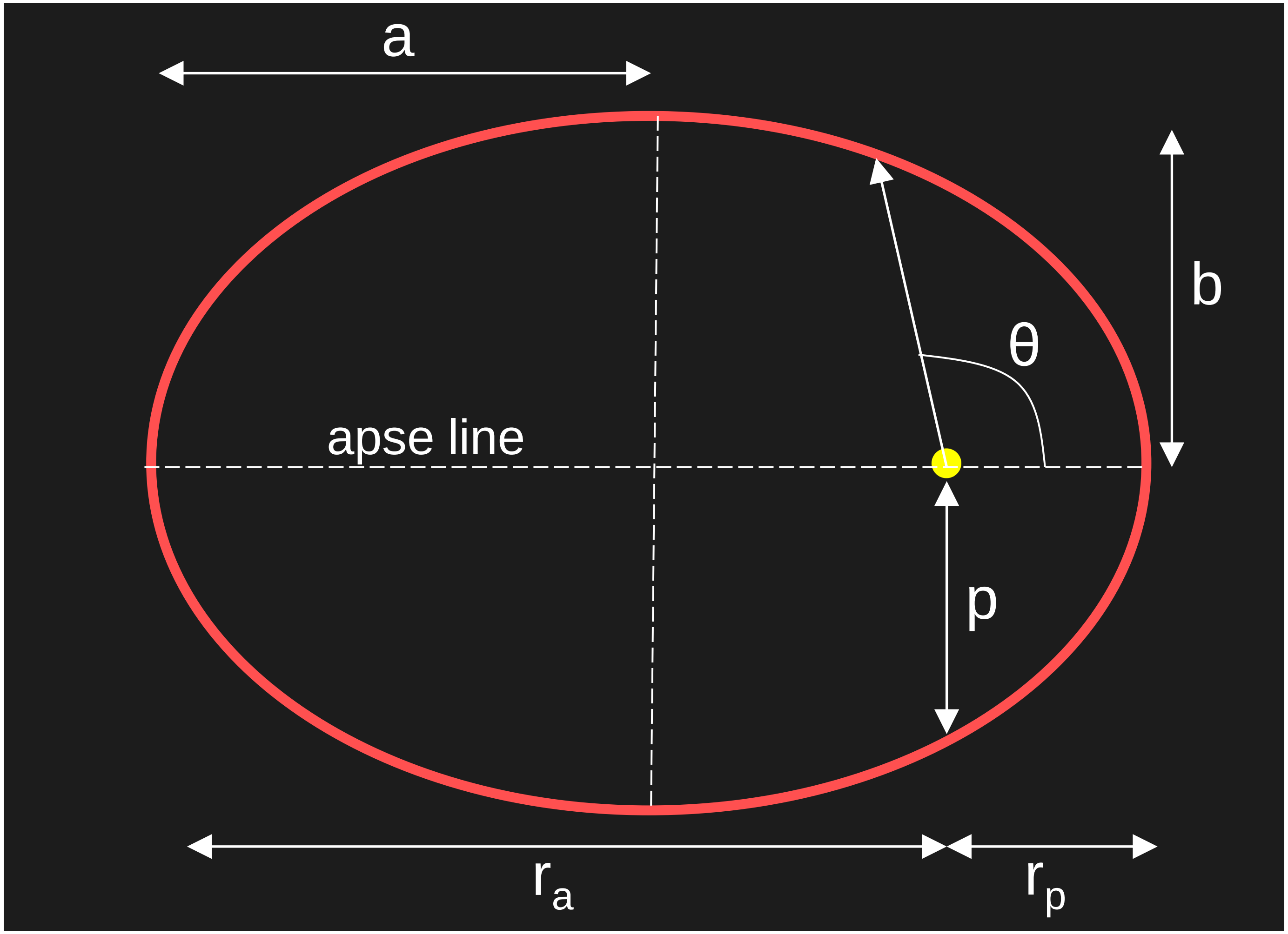
* A sidereal day, 23h56m4s, is the time it takes the Earth to complete one rotation relative to inertial space. A synodic day, 24h, is the time it takes the sun to apparently rotate once around the earth. They would be identical if the earth stood still in space.

A sidereal day



1 solar day= 1.0027378119113544 8 sidereal day

Geometry of the Elliptic Orbit



Elliptic Orbits (0<e<1)

$$r = \frac{h^2}{\mu} \frac{1}{1 + e\cos\theta}$$

The relative position vector remains bounded.

θ=0, minimum separation, **periapse**

$$r_p = \frac{h^2}{\mu(1+e)}$$

θ=π, greatest separation, **apoapse**

$$r_a = \frac{h^2}{\mu(1-e)}$$

θ=π/2, **semi-latus rectum** *p*

$$e = \frac{r_a - r_p}{r_a + r_p}$$

Energy of an Elliptical Orbit

$$\frac{v^2}{2} - \frac{\mu}{r} = E \qquad \frac{v_p^2}{2} - \frac{\mu}{r_p} = E_{perigee}$$

$h = v_p r_p$ See part 1

$$\frac{h^2}{2r_p^2} - \frac{\mu}{r_p} = E_{perigee}$$

$$r_p = \frac{h^2}{\mu(1+e)}$$

$$-\frac{1}{2}\frac{\mu^2}{h^2}(1 - e^2) = E_{perigee}$$

Link between energy and the other constants **h** and **e**!

$h = \sqrt{\mu a(1-e^2)}$ See next slide

$$-\frac{\mu}{2a} = E_{perigee}$$

Note: Angular Momentum

$$r = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}$$

Orbit equation

$$r = \frac{a(1-e^2)}{1+e\cos\theta}$$

Polar equation of an ellipse
(*a*, semimajor axis)

$$h = \sqrt{\mu a(1-e^2)}$$

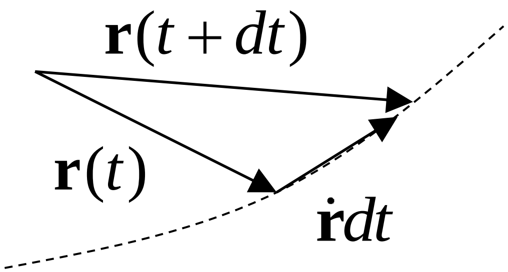
Velocity in an Elliptical Orbit

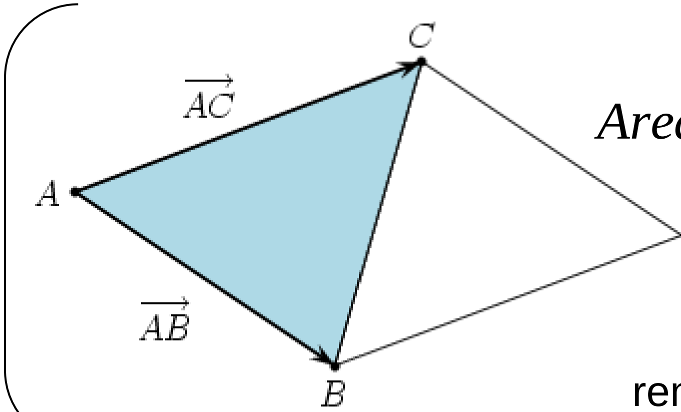
$$\frac{v^2}{2} - \frac{\mu}{r} = E \qquad -\frac{\mu}{2a} = E_{perigee}$$

$$\frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$$

$$v = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)}$$

Kepler's Second Law





$$Area = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$$

reminder

$$dA = \frac{1}{2}\left|\mathbf{r} \times \dot{\mathbf{r}}dt\right| = \frac{1}{2}\left|\mathbf{h}\right|dt = \frac{1}{2}hdt$$

$$\frac{dA}{dt} = \frac{h}{2} = \frac{1}{2}r^2\frac{d\theta}{dt} = \text{constant}$$

The line from the sun to a planet sweeps out equal areas inside the ellipse in equal lengths of time.

Kepler's Third Law

$$T = \frac{\text{enclosed area}}{dA/dt} = \frac{2\pi \text{ab}}{h}$$

$$h = \sqrt{\mu a(1-e^2)} \qquad b = a\sqrt{1-e^2}$$

$$T_{ellip} = 2\pi\sqrt{\frac{a^3}{\mu}}$$

The elliptic orbit period depends only on the semimajor axis and is independent of the eccentricity.

$$\frac{T_1^2}{T_2^2} = \frac{a_1^3}{a_2^3}$$

The squares of the orbital periods of the planets are proportional to the cubes of their mean distances from the sun.

Satellite in Elliptic Orbit

$$r_p = 354 + 6378 = 6732\,\text{km} \qquad r_a = 1447 + 6378 = 7825\,\text{km}$$

$$\Downarrow$$

$$e = \frac{r_a - r_p}{r_a + r_p} = 0.075, \quad a = \frac{r_a + r_p}{2} = 7278.5\,\text{km}$$

$$T = 2\pi\sqrt{\frac{a^3}{\mu}} = 6179.79\,\text{s} = 103\,\text{min}$$

$$v = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)} \rightarrow \begin{cases} v_p = 7.98\,\text{km/s} \\ v_a = 6.86\,\text{km/s} \end{cases}$$

GTO and GEO

For an orbit with a perigee at 320 km and an apogee at 35786 km, what is the velocity increment required to reach the geostationary orbit ?

$$v = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)}$$

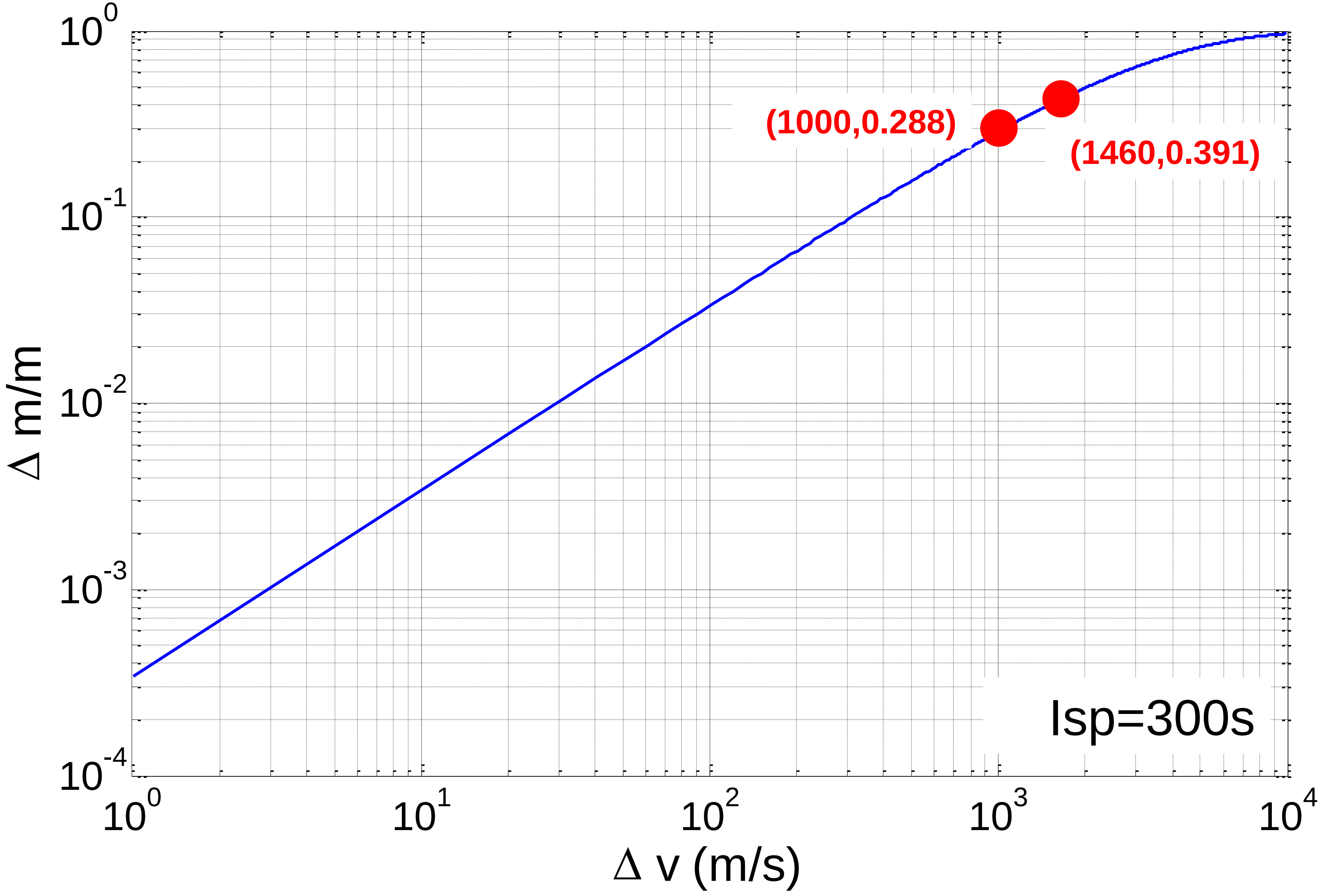
GTO and GEO

For an orbit with a perigee at 320 km and an apogee at 35786 km, what is the velocity increment required to reach the geostationary orbit ?

GTO	GEO
$a = \frac{r_a + r_p}{2} = 24430\,\text{km}$	$v_{circ} = \sqrt{\frac{398000}{35786 + 6378}}$
$v_p = 10.13\,\text{km/s}$	$= 3.07\,\text{km/s}$
$v_a = 1.61\,\text{km/s}$	

Answer: 1.46 km/s
(apogee motor)

GTO and GEO



(1000,0.288)
(1460,0.391)
Isp=300s
Δ m/m
Δ v (m/s)
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^0
10^1
10^2
10^3
10^4

Parabolic Orbits (e=1)

$$r = \frac{h^2}{\mu}\frac{1}{1+\cos\theta} \qquad \theta \to \pi,\ r \to \infty$$

$$\varepsilon_{parab} = -\frac{1}{2}\frac{\mu^2}{h^2}\left(1-e^2\right) = 0$$

The satellite has just enough energy to escape from the attracting body.

$$\varepsilon = \frac{v^2}{2} - \frac{\mu}{r}$$

$$v_{parab} = \sqrt{\frac{2\mu}{r}}$$

The satellite will coast to infinity, arriving there with zero velocity relative to the central body.

Escape Velocity, V_{esc}

11.2 km/s is the **second cosmic velocity**; i.e., the minimum velocity (theoretical velocity, $r = 6378$km) to escape the Earth.

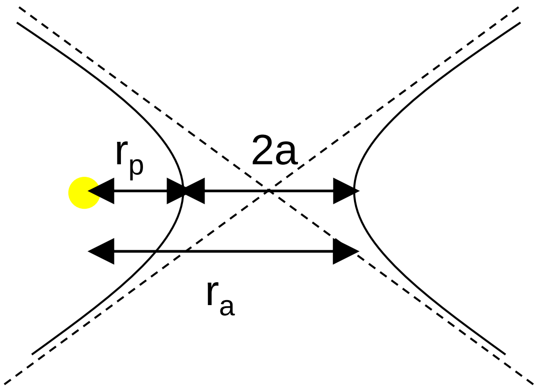
$$v_{circ} = \sqrt{\frac{\mu}{r}} \qquad v_{parab} = \sqrt{\frac{2\mu}{r}}$$

⇓

$$11.2\,\text{km/s} = \sqrt{2} \times 7.9\,\text{km/s}$$

Hyperbolic Orbits (e>1)

$$r = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}$$



$$r_p = \frac{h^2}{\mu(1+e)}$$

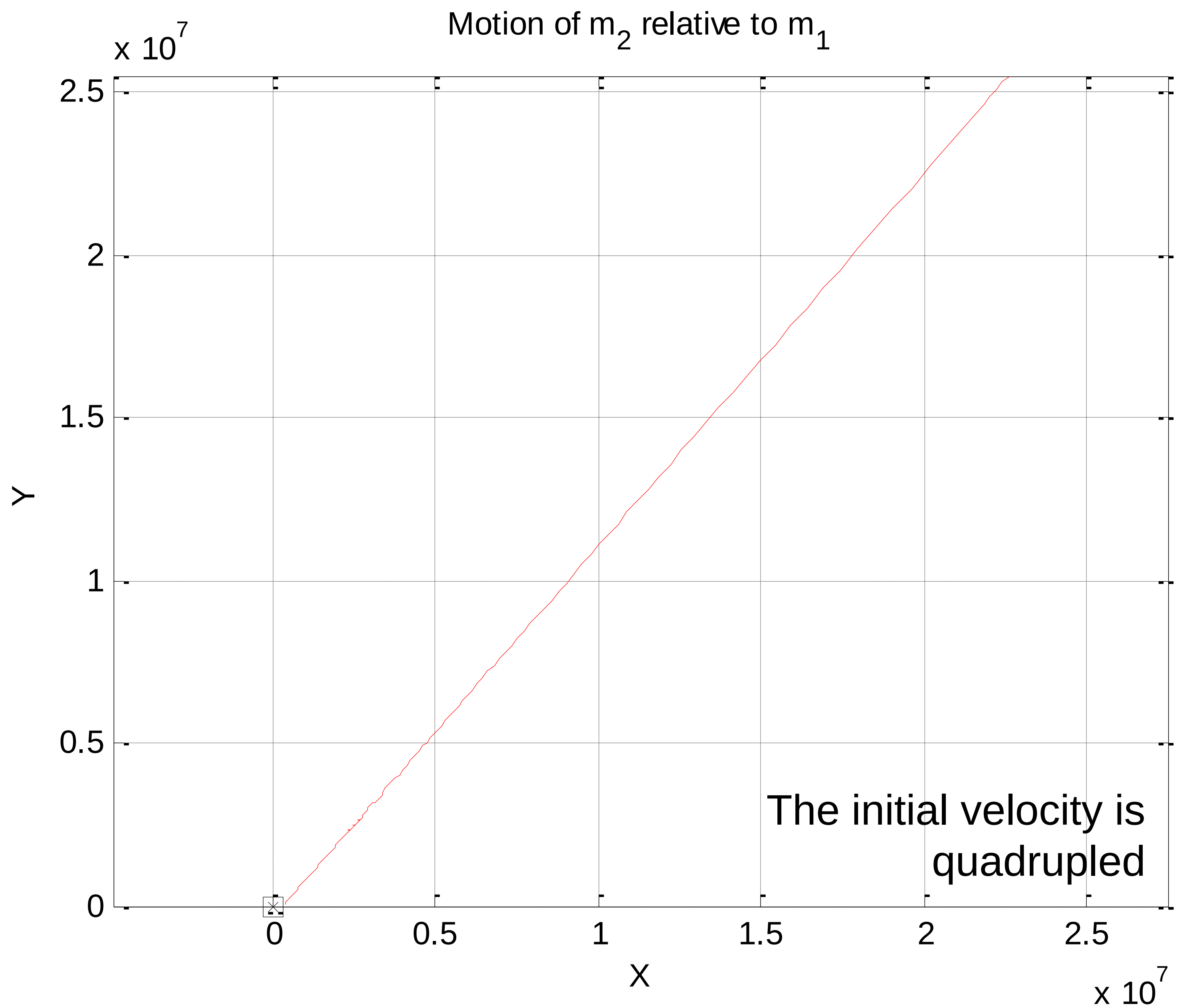
$$r_a = \frac{h^2}{\mu(1-e)} < 0$$

$$2a = -\left(r_a + r_p\right)$$

$$a = \frac{h^2}{\mu}\frac{1}{e^2-1}$$

$$\varepsilon = -\frac{1}{2}\frac{\mu^2}{h^2}\left(1-e^2\right)$$

$$\varepsilon_{hyper} = \frac{\mu}{2a} > 0$$



Motion of m2 relative to m1
x 10^7
2.5
2
1.5
1
0.5
0
Y
0
0.5
1
1.5
2
2.5
X
x 10^7
The initial velocity is quadrupled

C_3 Velocity

$$\varepsilon = \frac{\mu}{2a} = \frac{v^2}{2} - \frac{\mu}{r}$$

$$v_\infty = \sqrt{\frac{\mu}{a}}$$

Hyperbolic excess speed

$$\frac{v_\infty^2}{2} = \frac{v^2}{2} - \frac{\mu}{r}$$

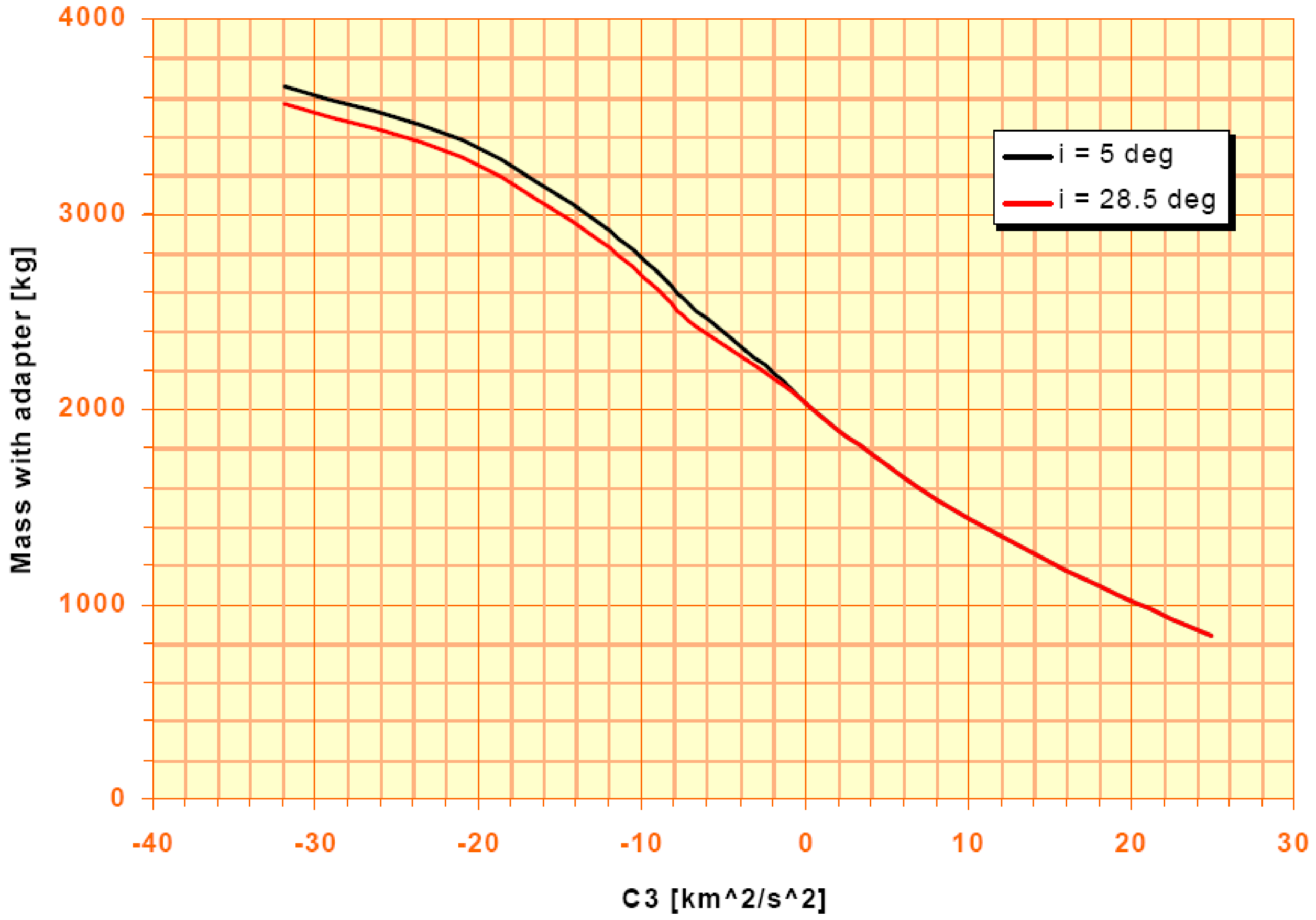
$$v^2 = v_\infty^2 + v_{esc}^2 = C_3 + v_{esc}^2$$

C_3 is a measure of the energy for an interplanetary mission:

16.6 km^2/s^2 (Cassini-Huygens)

8.9 km^2/s^2 (Solar Orbiter, phase A)

Soyuz ST v2-1b (Kourou Launch)



Proton

Table 2.9.1-1: Earth Escape Proton M Breeze M Missions

C3 Parameter (km^2/s^2)	Payload Systems Mass (kg)
-5	6270
-2	5890
0	5650
5	5090
10	4580
15	4110
20	3685
25	3295
30	2920
35	2575
40	2260
45	1990
50	1750
55	1525
60	1305
65	1120

C3 Parameter = $V^2 - 2\mu/R$.
Performance based on the use of 15255 mm PLF (standard).
At fairing jettison, FMHF shall be no more than 1135 W/m^2.
PSM includes LV adapter system mass.
PSM is calculated assuming a 2.33-sigma LV propellant margin.

What have we achieved so far ?

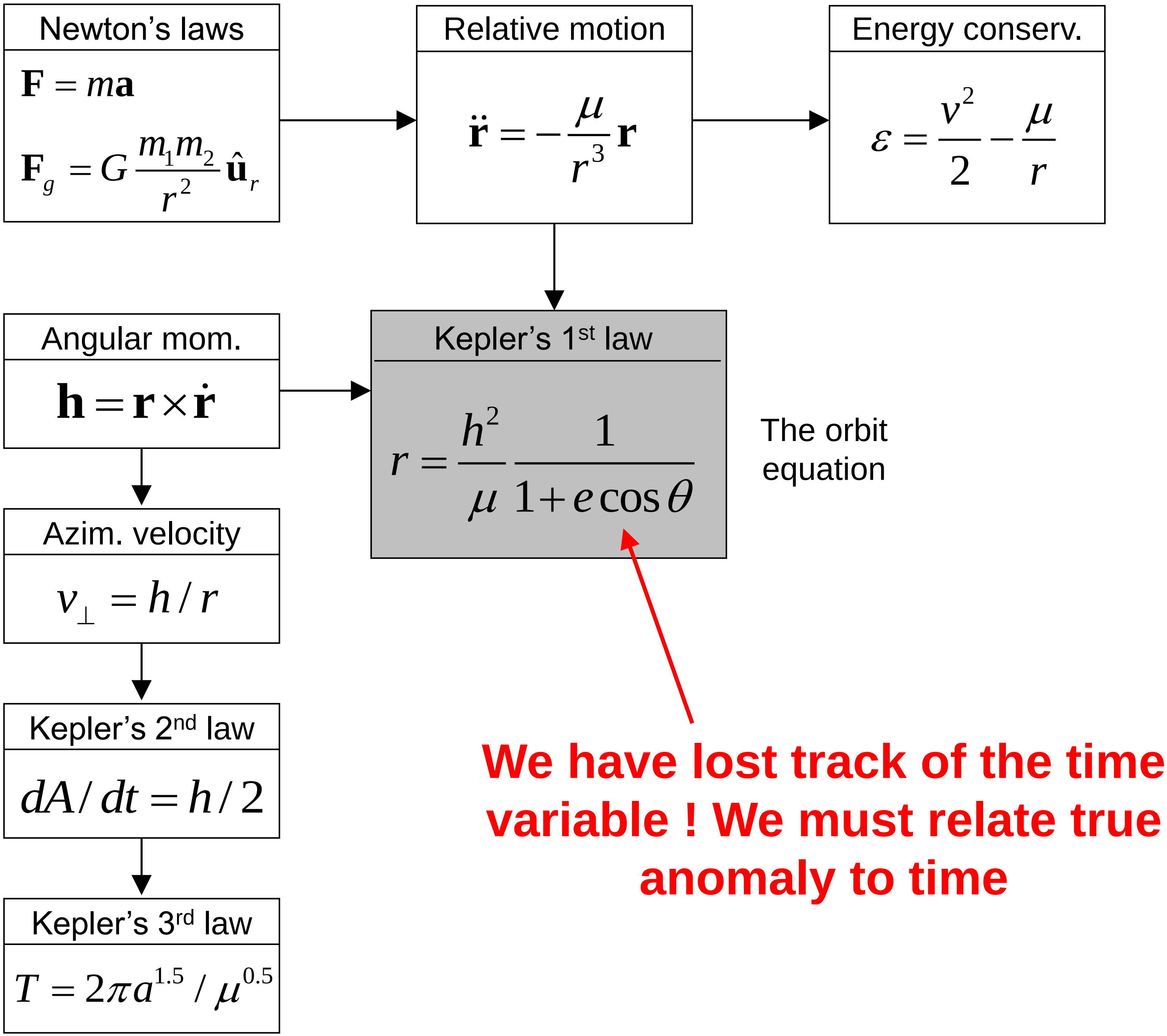
Closed-form solution from which we deduced Kepler's laws.

Analytic formulas for orbital energy, velocity and period.

Two-body propagator available in STK. Often used in early studies to perform trending analysis.

But …

We have lost track of the time variable !


Newton's laws
$\mathbf{F} = m\mathbf{a}$
$\mathbf{F}_g = G\frac{m_1 m_2}{r^2}\hat{\mathbf{u}}_r$
Relative motion
$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r}$
Energy conserv.
$\varepsilon = \frac{v^2}{2} - \frac{\mu}{r}$
Angular mom.
$\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}}$
Kepler's 1st law
$r = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}$
The orbit equation
Azim. velocity
$v_\perp = h/r$
Kepler's 2nd law
$dA/dt = h/2$
Kepler's 3rd law
$T = 2\pi a^{1.5} / \mu^{0.5}$
We have lost track of the time variable ! We must relate true anomaly to time

Time Since Periapsis

What is the time required to fly between any two true anomaly ?

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{h}{2} = \text{constant}$$

Kepler's second law

$$r = \frac{h^2}{\mu} \frac{1}{1 + e\cos\theta}$$

Orbit equation

$$\frac{\mu^2}{h^3} dt = \frac{d\theta}{\left(1 + e\cos\theta\right)^2}$$

$$\frac{\mu^2}{h^3}\left(t - t_p\right) = \int_0^\theta \frac{d\theta}{\left(1 + e\cos\theta\right)^2}$$

Sixth missing constant (t_p=0)

$$\frac{\mu^2}{h^3} t = \int_0^\theta \frac{d\theta}{\left(1 + e\cos\theta\right)^2}$$

Good news

$$\frac{\mu^2}{h^3}t = \int_0^{\theta} \frac{d\theta}{(1+e\cos\theta)^2}$$

Handbook of Mathematical Functions

With

Formulas, Graphs, and Mathematical Tables

Edited by
Milton Abramowitz and Irene A. Stegun

4.3.133

$$\int \frac{dz}{a+b\cos z} = \frac{2}{(a^2-b^2)^{\frac{1}{2}}} \arctan \frac{(a-b)\tan\frac{z}{2}}{(a^2-b^2)^{\frac{1}{2}}} \quad (a^2>b^2)$$

Circular Orbits

$$\frac{\mu^2}{h^3} t = \int_0^\theta \frac{d\theta}{(1 + e\cos\theta)^2}$$

⇩

$$\frac{\mu^2}{h^3} t = \int_0^\theta d\theta$$

⇩

$$t = \frac{h^3}{\mu^2}\theta = \frac{r^{3/2}}{\sqrt{\mu}}\theta = \frac{T}{2\pi}\theta$$

$$\rightarrow \theta = \frac{2\pi}{T} t$$

Obvious, because the angular velocity is constant.

Elliptic Orbits

Because the angular velocity of a spacecraft along an eccentric orbit is continuously varying, the expression of the angular position versus time is no longer trivial.

4.3.133

$$\int \frac{dz}{a+b\cos z} = \frac{2}{(a^2-b^2)^{\frac{1}{2}}} \arctan \frac{(a-b)\tan\frac{z}{2}}{(a^2-b^2)^{\frac{1}{2}}} \quad (a^2>b^2)$$

$$\frac{\mu^2}{h^3} t = \int_0^\theta \frac{d\theta}{(1+e\cos\theta)^2}$$

$$\overset{0<e<1}{=} \frac{1}{(1-e^2)^{3/2}} \left[\underbrace{2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right) - \frac{e\sqrt{1-e^2}\sin\theta}{1+e\cos\theta}}_{\text{Mean anomaly, } M} \right]$$

Mean anomaly, *M*

$$T_{ellip} = 2\pi\sqrt{\frac{a^3}{\mu}}$$

$$h = \sqrt{\mu a(1-e^2)}$$

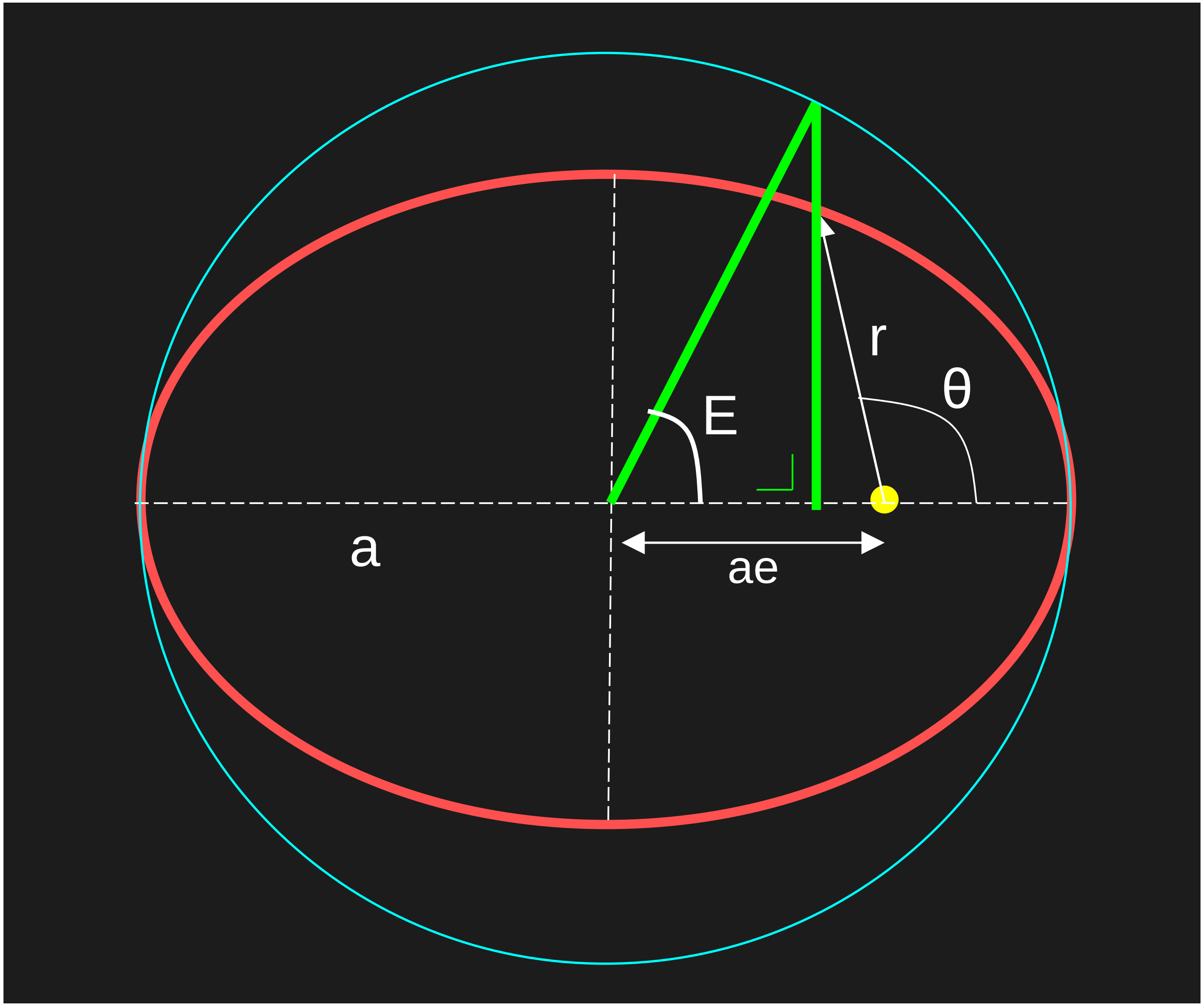
$$M = \frac{\mu^2}{h^3}(1-e^2)^{3/2} t = \frac{2\pi}{T} t = nt$$

Mean Anomaly Is Related to Time

For circular orbits, the mean M and true anomalies θ are identical.

For elliptic orbits, the mean anomaly represents the angular displacement of a fictitious body moving around the ellipse at the constant angular speed n.

Eccentric Anomaly Is Related to Position



Eccentric Anomaly: Relation with Mean Anomaly ?

$$\cos^2 E + \sin^2 E = 1$$

$$a\cos E = ae + r\cos\theta$$ Graph

$$r = a\frac{1-e^2}{1+e\cos\theta}$$ L2

$$\Rightarrow \cos E = \frac{e+\cos\theta}{1+e\cos\theta} \Rightarrow \sin E = \frac{\sqrt{1-e^2}\sin\theta}{1+e\cos\theta}$$

trig. id.

$$\cos E = \frac{e+\cos\theta}{1+e\cos\theta}$$

$$\sin E = \frac{\sqrt{1-e^2}\sin\theta}{1+e\cos\theta}$$

$$\Rightarrow \tan^2\frac{E}{2} = \frac{1-\cos E}{1+\cos E} = \frac{1-e}{1+e}\frac{1-\cos\theta}{1+\cos\theta} = \frac{1-e}{1+e}\tan^2\frac{\theta}{2}$$

$$\Downarrow$$

$$E = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right)$$

Kepler's Equation

$$M = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right) - \frac{e\sqrt{1-e^2}\sin\theta}{1+e\cos\theta} \qquad E = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right)$$

$$M = nt = E - e\sin E$$

It relates time, in terms of $M=nt$, to position, in terms of E, $r=a(1-e.\cos E)$.

Usefulness of Kepler's Equation

θ and orbit are given

⇩

$$E = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right)$$

⇩

$$M = E - e\sin E$$

⇩

$$t = \frac{M}{2\pi}T$$

Practical application:

Determine the time at which a satellite passes from sunlight into the Earth's shadow (the location of this point is known from the geometry).

Example

A geocentric elliptic orbit has a perigee radius of 9600 km and an apogee radius of 21000 km. Calculate the time to fly from perigee to a true anomaly of 120°.

$$e = \frac{r_a - r_p}{r_a + r_p} = \frac{21000 - 9600}{21000 + 9600} = 0.37255$$

$$E = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right) = 1.7281\,\text{rad}$$

$$M = E - e\sin E = 1.3601\,\text{rad}$$

$$T = 2\pi\sqrt{\frac{a^3}{\mu}} = 2\pi\sqrt{\left(\frac{r_p + r_a}{2}\right)^3 / \mu} = 18834\,\text{s}$$

$$t = \frac{M}{2\pi}T = \frac{1.3601}{2\pi}18834 = 4077\,\text{s} = 1\text{h}07\text{m}57\text{s}$$

Usefulness of Kepler's Equation

t is given

↓

$$M = \frac{2\pi t}{T}$$

↓

$$M = E - e \sin E$$

↓

Transcendental equation !!!
(with a unique solution)

Practical application:

Perform a rendez-vous with the ISS (ATV, STS, Soyuz, Progress).

Numerical Solution: Newton-Raphson

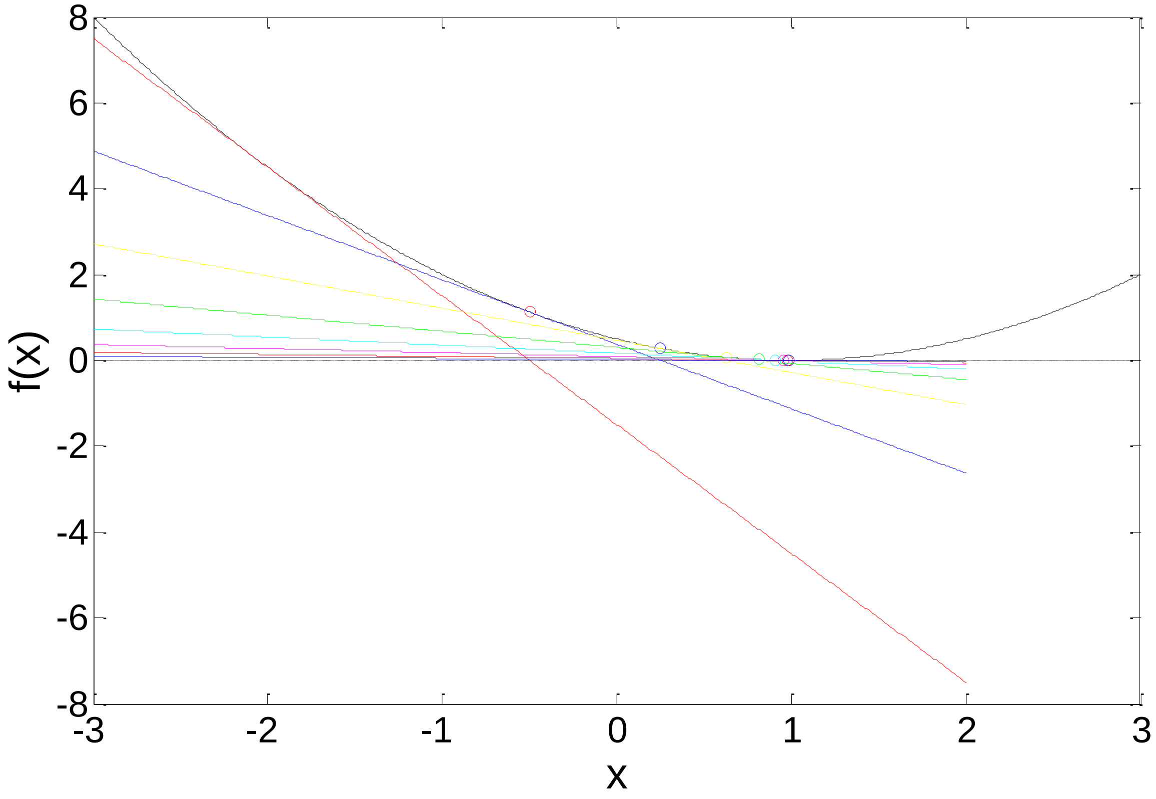
Algorithm for finding approximations to the zeros of a nonlinear function.

Recursive application of Taylor series truncated after the first derivative.

The initial guess should be close enough to the actual solution.

Numerical Solution: Newton-Raphson

Example: find the zero of $f(x)=0.5(x-1)^2$



Analytic Solution: Lagrange

$$E = M + \sum_{n=1}^{\infty} a_n e^n$$

$$a_n = \frac{1}{2^{n-1}} \sum_{k=0}^{floor(n/2)} (-1)^k \frac{1}{(n-k)!k!} (n-2k)^{n-1} \sin\left[(n-2k)M\right]$$

Convergence if e<0.663.

For small values of the eccentricity a good agreement with the exact solution is obtained using a few terms (e.g., 3).

Analytic Solution: Bessel Functions

$$E = M + \sum_{n=1}^{\infty} \frac{2}{n} J_n(ne) \sin nM$$

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(n+k)!k!} \left(\frac{x}{2}\right)^{n+2k}$$

Convergence for all values of the eccentricity less than 1.

Prediction of the Position and Velocity

If the position and velocity $\boldsymbol{r}_0$ and $\boldsymbol{v}_0$ of an orbiting body are known at a given instant t_0, how can we compute the position and velocity $\boldsymbol{r}$ and $\boldsymbol{v}$ at any later time t ?

Concept of f and g function and series:

$$\mathbf{r}(t) = f(t, t_0, \mathbf{r}_0, \mathbf{v}_0)\mathbf{r}_0 + g(t, t_0, \mathbf{r}_0, \mathbf{v}_0)\mathbf{v}_0$$

Prussing and Conway

$$f = 1 - \frac{a}{r_0}\left[1 - \cos(E - E_0)\right]$$

J.E. Prussing, B.A. Conway, *Orbital Mechanics*, Oxford University Press

$$g = (t - t_0) - \sqrt{\frac{a^3}{\mu}}\left[(E - E_0) - \sin(E - E_0)\right]$$

Prediction of the Position and Velocity

Some form of Kepler's equation must still be solved by iteration. However, Gauss developed a series expansion in the elapsed time parameter t-t_0, and there is no longer the need to solve Kepler's equation:

$$\mathbf{r}(t) = f(t, t_0, \mathbf{r}_0, \mathbf{v}_0)\,\mathbf{r}_0 + g(t, t_0, \mathbf{r}_0, \mathbf{v}_0)\,\mathbf{v}_0$$

$$f = \left[1 - \frac{\mu}{2r_0^3}(t - t_0)^2 + \frac{\mu}{2}\frac{\mathbf{r}_0 . \mathbf{v}_0}{r_0^5}(t - t_0)^3 + \ldots\right]$$

$$g = \left[(t - t_0) - \frac{\mu}{6r_0^3}(t - t_0)^3 + \frac{\mu}{4}\frac{\mathbf{r}_0 . \mathbf{v}_0}{r_0^5}(t - t_0)^4 + \ldots\right]$$

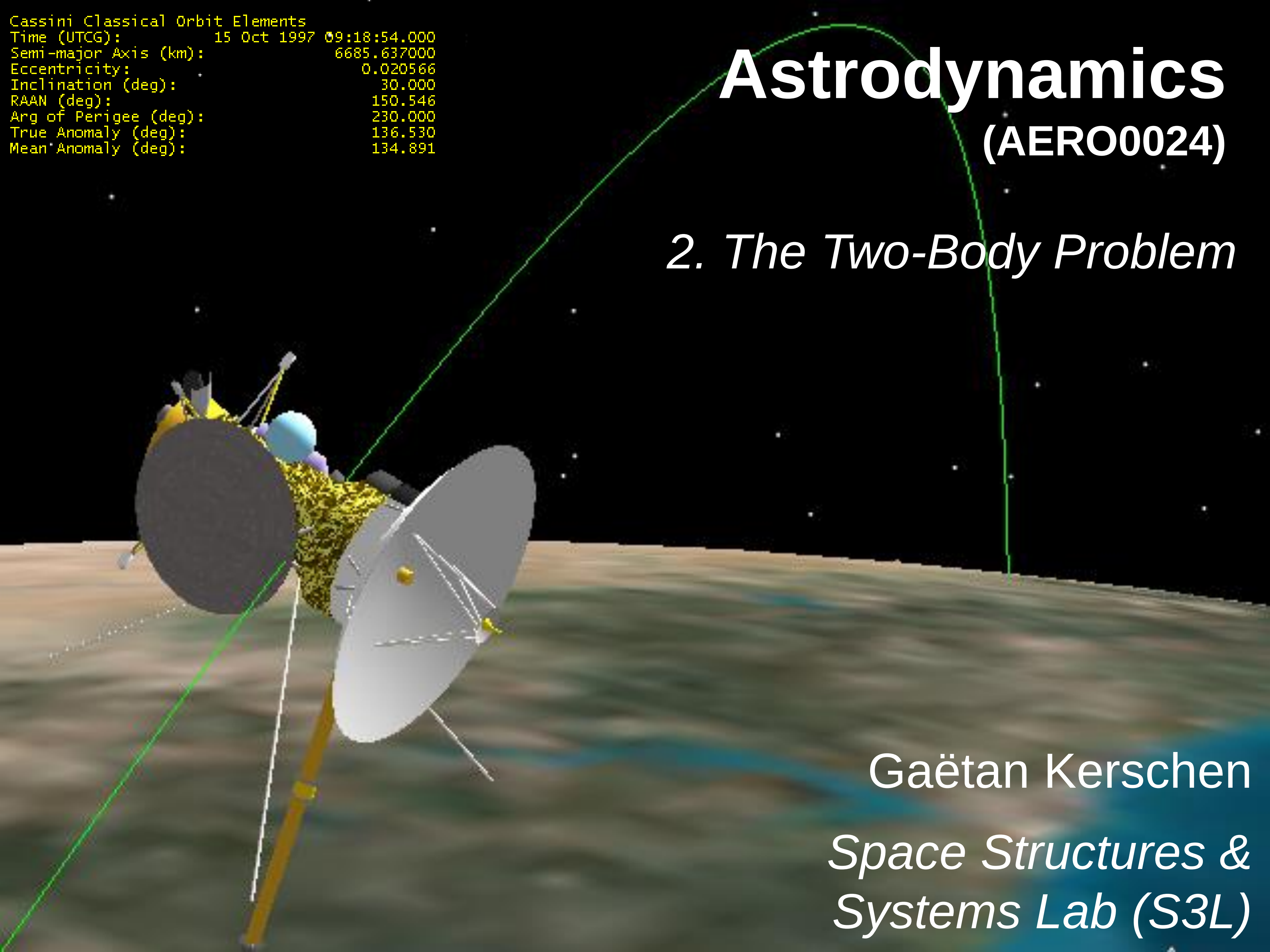
Did you Know ?

Compactness of the solar system: measured by the ratio of the distance *a* of a planet from the Sun to the radius R of the Sun.

$$\frac{a}{R} \sim 200$$

Compactness of the hydrogen atom: measured by the ratio of the distance *a* of an electron from the nucleus to the radius R of the nucleus.

$$\frac{a}{R} \sim 5e4$$



Astrodynamics

(AERO0024)

2. The Two-Body Problem

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